Centro de Investigación Operativa

I-2007-19

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July 2007

ISSN 1576-7264

Depósito legal A-646-2000

TRABAJOS I+D

Centro de Investigación Operativa

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On Stochastic Dynamic Programming for solving large-scale planning problems under uncertainty

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Statement of Scope and purpose
It has long been recognized that traditional deterministic optimization is not suitable for capturing the truly dynamic behavior of most real-life problems, given the uncertainty of the parameters that represent information about the future. Many of these problems, planning under uncertainty, have logical constraints that require 0–1 variables for their formulation. The solution of this type of problems can be performed via Stochastic Integer Programming by using scenario tree analysis. Given the dimensions of the Deterministic Equivalent Model (DEM) of the stochastic problem, some kinds of decomposition approaches can be considered by exploiting the structure of the models. Traditional decomposition schemes, such as Benders and Lagrangean approaches, do not seem to provide the solution for large scale problems (mainly in the cardinality of the scenario tree) in affordable computing effort. In this work we present a Stochastic Dynamic Programming approach, specially suited for exploiting the structure of the scenario tree and, thus, very amenable for solving very large-scale DEMs. A tactical production planning problem is used as a pilot case, whose structure is not exploited by the proposed approach, so that it has general applicability.

Abstract
The Stochastic Dynamic Programming approach that we present utilizes the scenario tree in a back-to-front scheme. It obtains the solution of the multi-period stochastic problems related to the subtrees whose root nodes are the starting nodes (i.e., scenario groups) in each given stage along the time horizon. Each subproblem considers the effect of the stochasticity of the uncertain parameters from the periods of the given stage, by using curves that estimate the expected future value (EFV) of the objective function. Each subproblem is solved for a set of reference levels of the variables that also have nonzero elements in any of the previous stages besides the given stage. An

appropriate sensitivity analysis of the objective function for each reference level of the linking variables allows us to estimate the *EFV* curves for the scenario groups from the previous stages, until the curves for the first stage are computed. An application of the scheme to the production planning problem with logical constraints is presented. The aim of the problem consists of obtaining the tactical production planning over the scenarios along the time horizon. The expected total cost is minimized to satisfy the product demand. Some computational experience is reported. The proposed approach compares favorably for very large-scale instances with a state-of-the-art optimization engine.

Keywords: Stochastic dynamic programming, scenario tree, mixed 0–1 model, tactical production planning.

* This research has been partially supported by grant GRUPOS79/04 from the Generalitat Valenciana, Spain.

1 Introduction

Let the following dynamic multi-linking constraint deterministic program,

$$\begin{aligned} \min \quad & \sum_{t\in\mathcal{T}} c_t x_t \\ \text{s.t.} \quad & \sum_{t\in\mathcal{T}:t\leq\tau} A_\tau^t x_t = b_\tau \quad && \forall \tau \in \mathcal{T} \\ & x_t \in \mathcal{X} && \forall t \in \mathcal{T}, \end{aligned} \tag{1.1}$$

where $\mathcal{T}$ is the set of periods in a given time horizon, x_t is the vector of the variables related to time period t, c_t is the row vector of the objective function coefficients, A_τ^t is the constraint matrix related to time period τ for the variables x_t, $\mathcal{X}$ is the set of feasible solutions including the nonnegativity and the 0–1 values for the variables, and b_τ is the right-hand-side (*rhs*) vector for the constraints related to time period τ, for $\tau \in \mathcal{T}$, all with the appropriate dimensions. A particular case for variables with nonzero elements in constraints related to two time periods, not necessarily consecutive, is presented in [10] for the case of multi-level product supplying with a transportation time interval greater than two periods. A typical case with variables having nonzero elements in constraints related to two consecutive time periods is the stock of goods in one period to be used in the next one, see [2, 12], among many others. See also section 5 below.

However, very frequently some of the parameters (mainly, the objective function coefficients and the *rhs*) are not known with certainty when the decision is to be made. Given today's state-of-the-art optimization tools, deterministic mixed integer programs (MIP) should not present major difficulties for solving moderate size cases, at least. Moreover, it has long been recognized that traditional deterministic optimization is not suitable for capturing the truly dynamic behavior of most real-life applications. The main reason is that such applications involve, as previously stated, data uncertainties which arise because information that will occur in subsequent decision stages is not available to the decision maker when the decision must be made. For our purposes it suffices to consider the uncertainty of the vectors b and c. The stochastic problem will be treated by using a scheme, such that the parameters uncertainty is visualized by a scenario tree.

Stochastic Integer Programming has a broad application field and is flourishing in such sectors as finance, telecommunications, hydrothermal electric generation, oil, hydrocarbon and chemical supplying, transformation and distribution logistics, strategic and tactical production planning, supply chain management, site location in addition to other sectors. See particularly the books [14, 24, 26, 28], among others. Given the dimensions of the cases, some kinds of decomposition approaches are considered, most of them based on Benders, Lagrangean and Branch-and-Fix Coordination decomposition schemes for the structured mixed 0–1 *Deterministic Equivalent Model* (*DEM*); for recommended text books on Stochastic Programming, see [5, 13, 15], among others.

Moreover, decomposition schemes based on the above approaches do not seem to provide the solution for large-scale problems (mainly, in the scenario tree) in reasonable computing effort. Alternatively, some kinds of Stochastic Dynamic Programming (*SDP*) has been used for solving water resources management problems, see in a different context [6, 7, 8, 11, 16]. See in [18, 19] the work inspiring this paper.

The purpose of this paper is to present an *SDP* approach that utilizes a scenario tree *back-to-front* scheme, and obtains the solution of the multi-period stochastic mixed 0–1 subproblems related to the subtrees whose root nodes are the starting nodes (i.e., scenario groups) in each stage along the time horizon. Each subproblem considers the effect of the stochasticity of the uncertain parameters from the given stage, by using curves that give the *Expected Future Value* (*EFV*) of the objective function. Each subproblem is solved for a representative set of *reference levels* of the linking variables between the previous stages and a given one. An appropriate sensitivity analysis of the objective function for a set of *reference levels* of the variables allows for estimating the *EFV* curves for the scenario groups from the previous stages, until the curves for the first stage are computed. In this way, the *EFV* curves of the variables for the implementable time periods (i.e., stage 1) are obtained by considering all scenarios, but without being subordinated to any of them. The solution to be obtained for the first stage considers the influence of the scenarios by using the *EFV* curves that have been obtained. So, the original mixed 0–1 problem is broken down into as many subproblems as subtrees we have in the scenario tree where the roots are the starting nodes of each stage. The 0–1 variables and the continuous variables are allowed for at any stage in the time horizon. The scope of this paper only considers the continuous linking variables.

Additionally, an application of the proposed methodology is considered for the tactical production planning problem. This application will be used as a pilot case to assess the validity of the scheme proposed in the paper. The goal consists of determining the production and stocking of a set of products to satisfy an uncertain demand along a time horizon, at a minimum expected cost, subject to resource availability and logical constraints, among others. The uncertain parameters are the product cost and demand and the resource availability along the time horizon. The uncertainty is represented by a scenario tree. Two versions of the *DEM* are presented as multistage mixed 0-1 programs, where the 0-1 variables are the tactical variables (decisions on products and lot sizing), and the continuous variables represent the production, stock and lost demand at each time period. The proposed approach compares favorably with a state-of-the-art optimization engine for very large-scale *DEM* instances (over one million 0-1 variables and three million continuous variables).

The remainder of the paper is organized as follows. Section 2 formally introduces the problem to be solved. Section 3 presents the *SDP* algorithmic framework for problem solving, and introduces the concepts to be used for obtaining the *EFV* curves. Section 4 proposes the scheme for computing them. Section 5 presents the production planning problem. Section 6 reports on the computational experience. Finally, section 7 concludes.

2 Problem statement

Let the scenario tree shown in Figure 1 represent the stochasticity of the problem to be dealt with. Each node in the figure represents a point in time where a decision can be made. Once a decision is made, some contingencies can arise (in this example the number of contingencies is three for time period $t = 5$), and information related to these contingencies is available at the beginning of the period. Each root-to-leaf path in the tree represents one specific scenario and corresponds to one realization of the whole set of the uncertain parameters. Each node in the tree can be associated with a scenario group, such that two scenarios belong to the same group from a given time period provided that they have the same realizations of the uncertain parameters up to the period. Accordingly with the non-anticipativity principle [5, 21], the scenarios that belong to the same group at a given time period should have the same value for the related variables with the time index up to the period.

Let the following notation related to the scenario tree be used throughout the paper:

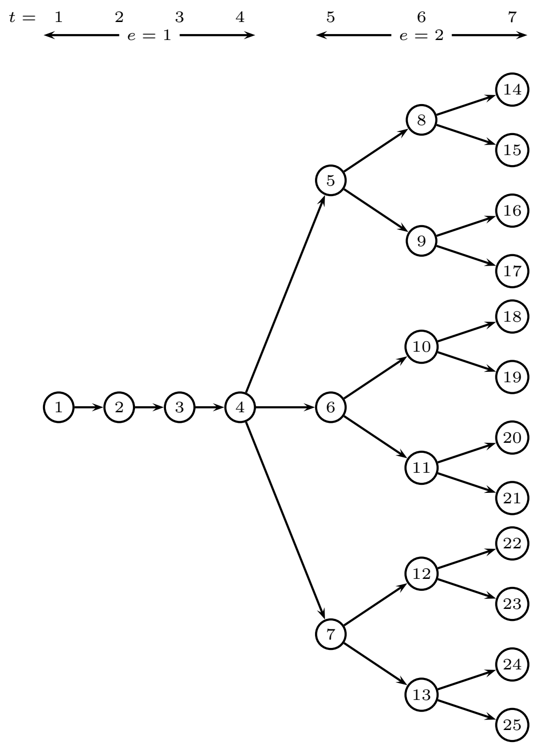
Ω**,** set of scenarios, consecutively numbered. For instance, the path $\{1, 2, \dots, 5, 8, 14\}$ gives a scenario, that is customarily named scenario 14.

$\mathcal{G}$**,** set of scenario groups, consecutively numbered.

$\mathcal{G}_t$**,** set of scenario groups in time period t, for $t \in \mathcal{T}$ ($\mathcal{G}_t \subseteq \mathcal{G}$).

Ω_g**,** set of scenarios in group g, such that the scenarios that belong to the same group are identical in all realizations of the uncertain parameters up to period $t(g)$, for $g \in \mathcal{G}$ ($\Omega_g \subseteq \Omega$).

$\mathcal{K}_g$**,** set of scenario groups $\{k\}$ such that $\Omega_g \subseteq \Omega_k$, for $g \in \mathcal{G}$ ($\mathcal{K}_g \subset \mathcal{G}$). That is, set of scenario groups (one for each time period), such that the set of scenarios in group



$$\Omega = \Omega_1 = \{14, 15, \ldots, 25\}$$
$$\mathcal{G}_5 = \{5, 6, 7\};\ t(12) = 6$$
$$\mathcal{K}_{13} = \{7, 4, \ldots, 1\}$$
$$\mathcal{A}^2 = \{5, 6, 7\}; \mathcal{A}_3 = \{5, 6, 7\}; \rho(12) = 7$$
$$\mathcal{C}_5 = \{5, 8, 9, 14, \ldots, 17\}$$

Figure 1: Multi-period scenario tree

g is included in the set of scenarios that belong to each of the groups $\{k\}$. Notice that the (sole) ancestor path from the associated node with scenario group g to the root node in the given scenario tree passes through all the associated nodes with the scenario groups in $\mathcal{K}_g$. From now on we will indistintly use nodes in the scenario tree and scenario groups.

$\rho(g)$, immediate ancestor node of node g in the scenario tree. Note $\rho(g) \in \mathcal{K}_g$.

$t(g)$, time period for scenario group g, for $g \in \mathcal{G}$. Note $g \in \mathcal{G}_{t(g)}$.

$\mathcal{E}$, set of stages in the given time horizon. Note: A stage is included by a set of consecutive time periods.

$\mathcal{G}^e$, set of scenario groups from stage e, for $e \in \mathcal{E}$.

$\mathcal{A}^e$, set of scenario groups associated with the root nodes from stage e, for $e \in \mathcal{E}$. ($\mathcal{A}^e \subseteq \mathcal{G}^e$)

$\mathcal{A}_g$, set of successor nodes of node g such that they belong to set A^{e+1}, for $g \in \mathcal{G}^e, e = 1, \ldots, |\mathcal{E}| - 1$.

$\mathcal{C}_a$, set of successor nodes in the subtree whose root is node a such that they belong to set $\mathcal{G}^e$, including itself, for $a \in \mathcal{A}^e, e \in \mathcal{E}$.

$\mathcal{S}_g$, set of successor nodes in the subtree whose root is node g, including itself, for $g \in \mathcal{G}$.

$\mathcal{P}_g^k$, set of successor nodes in the path from node g to node k, for $k \in \mathcal{S}_g, g \in \mathcal{G}$.

In order to present the stochastic version of program (1.1), let the following notation be used for the variables and the uncertain parameters:

Variables:

x^g, variables vector under scenario group g, for $g \in \mathcal{G}$. It replaces the variable $x_{t(g)}$ in the deterministic model.

Uncertain parameters:

c^g, vector of the objective function coefficients for the variables x^g under scenario group g, for $g \in \mathcal{G}$. It replaces the vector $c_{t(g)}$ in the deterministic model.

b^g, *rhs* of the constraint system under scenario group g, for $g \in \mathcal{G}$. It replaces the vector $b_{t(g)}$ in the deterministic model.

Let also w^g denote the weight factor representing the likelihood that is associated with scenario group g, for $g \in \mathcal{G}$.

The *Deterministic Equivalent Model (DEM)* of the multi-period stochastic program with complete recourse for optimizing the expected objective function value over the set of scenarios for program (1.1) has the following so-called *compact* representation,

$$\begin{aligned}
\min \quad & \sum_{g\in\mathcal{G}} w^g c^g x^g \\
\text{s.t.} \quad & \sum_{k\in\{g\}\bigcup\mathcal{K}_g} A_{t(g)}^{t(k)} x^k = b^g \quad \forall g\in\mathcal{G} \qquad (2.1)\\
& x^g \in \mathcal{X} \qquad\qquad\qquad \forall g\in\mathcal{G}.
\end{aligned}$$

3 Algorithmic framework for using the *EFV* curves

The proposed scheme consists of computing the *EFV* curves for the scenario groups at the previous stages of a given one. The curves estimate the impact of the decisions to be made in a given stage on the objective function value related to the future stages.

The strategy for computing the curves is based on a recursive procedure for using the state variables. It considers sets of *reference levels* for the linking variables between stages, and obtains the *EFV* curves for the scenario groups at any stage, say, $e' < e$ for $e = |\mathcal{E}|, \ldots, 2$ by using the curves obtained for stage e. The subproblems attached to the subtrees given by node set $\mathcal{C}_a$, $\forall a \in \mathcal{A}^e$, for any $e = 2, \ldots, |\mathcal{E}|$ share the same *reference levels* of the linking variables from node k such that $a \in \mathcal{A}_k$. So, the curves are obtained as the average from the curves that are computed by using the scenario subtrees at stage e.

An ad-hoc sensitivity analysis to be performed by truncating the Taylor series expansion of the objective function around the values of the linking variables in the given set of *reference levels* provides the information for computing the *EFV* curves. By using this *back-to-front* mechanism down to the second stage, the *EFV* curves are obtained for the first stage. Note: By construction, there is only one scenario path in the first stage.

Let $\mathcal{Z}^{e-1,k}$ denote the set of the *reference levels* for the values of the linking variables x^k for $k \in \mathcal{K}_a, a \in \mathcal{A}^e$ between the time periods under the scenario groups from the previous stages and stage e, for $e = |\mathcal{E}|, \ldots, 2$. A multi-period complete recourse stochastic subproblem, say, $\mathcal{P}^{az}$ must be solved for the scenario subtree headed by node, say, a for $a \in \mathcal{A}^e$ in stage e (being C_a the related set of nodes) for each *reference level* $z \in \mathcal{Z}^{e-1,k}$ of the variables' vectors x^k $\forall k \in \mathcal{K}_a$. For this purpose, let $\overline{x}^{kz}$ denote the value of the vector x^k. The objective function is also included by the *EFV* curves associated with the scenario groups, say, g for $g \in \mathcal{C}_a$. We propose to estimate the curve, say, $\lambda(x^k)$ as an upper envelop of a finite family of linear functions, continuous and convex but not differentiable everywhere. The program to solve can be expressed,

$$
\begin{aligned}
\mathcal{P}^{az}: \quad \min \quad & \sigma^{az} = \sum_{g \in \mathcal{C}_a} w^g \left(c^g x^g + \lambda^g \right) \\
\text{s.t.} \quad & \sum_{k \in \{g\} \bigcup \mathcal{K}_g} A^{t(k)}_{t(g)} x^k = b^g && \forall g \in \mathcal{C}_a \\
& x^k = \overline{x}^{kz} && \forall k \in \mathcal{K}_a \\
& x^g \in \mathcal{X} && \forall g \in \mathcal{C}_a \\
& \lambda^g \geq \mu^{gz'} + \pi^{gz'} x^g && \forall z' \in \mathcal{Z}^{e'g}, g \in \mathcal{C}_a, e' \in \mathcal{E} | e < e',
\end{aligned}
\tag{3.1}
$$

where $\overline{x}^{kz}$ is the value of x^k in a given *reference level* z, and the following variable and coefficients are to be obtained according to the scheme presented in section 4:

λ^g, variable that takes the value of the *EFV* curve for the variables' vector x^g.

μ^{gz}, constant term for the segment related to *reference level* z, for $z \in \mathcal{Z}^{eg}$.

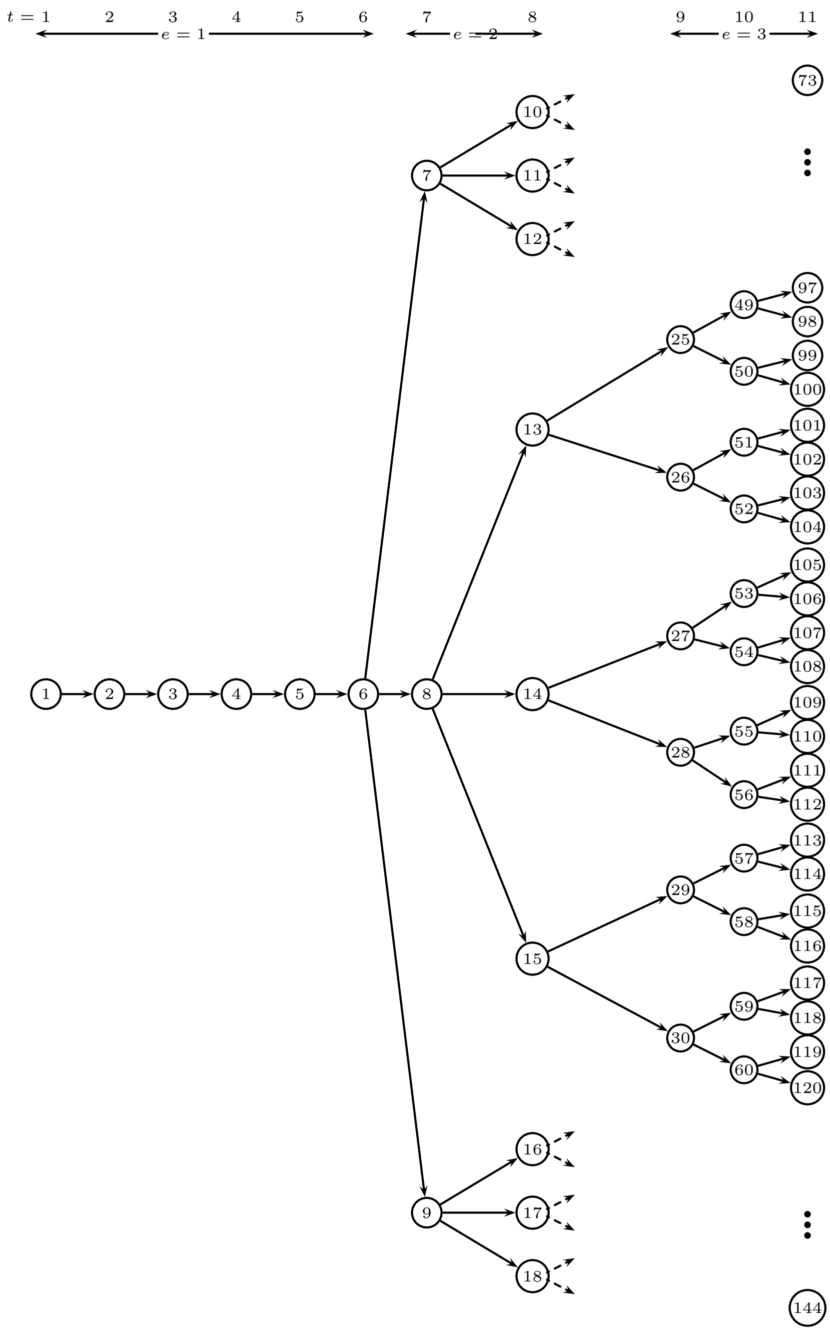
π^{gz}, row vector that gives the marginal *EFV*, due to the linking variables vector x^g, for *reference level* z.

So, the aim consists of solving a set of multi-period stochastic subproblems $\{\mathcal{P}^{az} : z \in \mathcal{Z}^{e-1,k}, k \in \mathcal{K}_a, a \in \mathcal{A}^e\}$ for each stage $e = |\mathcal{E}|, \ldots, 2$. The uncertainty of the parameters is represented by the scenario groups associated with the nodes in the subtree whose root node is node a. By using the appropriate *back-to-front* scheme, see below, the *EFV* curves are obtained for the scenario groups from stage $e = 1$. Notice that the subproblems $\mathcal{P}^{az}$ for a given $a \in \mathcal{A}^e$ share the same parameters, being the fixing $x^k = \overline{x}^{kz}, z \in \mathcal{Z}^{e-1,k}$ the only difference in their respective models.

Figure 2 shows a case for $|\mathcal{E}| = 3$ stages, where no *EFV* curves are for the nodes from stage $e = 3$ (i.e., $G^3 = \{73, \ldots, 96, 97, \ldots, 120, 121, \ldots, 144\}$). Notice that the optimization of the multi-period mixed 0–1 subproblems associated with the node sets $\mathcal{C}_{19}, \ldots, \mathcal{C}_{25} = \{25, 49, 50, 97, \ldots, 100\}$ $, \ldots, \mathcal{C}_{30}, \ldots, \mathcal{C}_{36}$ (i.e., stage e=3) can be performed for a set of the *reference levels* of the variables associated with the nodes from $\mathcal{G}^1 \bigcup \mathcal{G}^2 = \{1, \ldots, 18\}$. As a result, a set of *EFV* curves is obtained for the nodes 1 to 18. Notice that the optimization of the multi-period mixed 0–1 subproblems associated with the following subtrees $\mathcal{C}_7 = \{7, 10, 11, 12\}$, $\mathcal{C}_8, \mathcal{C}_9$ (i.e., stage e=2) can be performed for a set of the *reference levels* of the variables associated with the nodes 1 to 6. As a result, a set of *EFV* curves is obtained for the nodes from $\mathcal{G}^1$, see Figure 3.

4 On computing the *EFV* curves

The proposed scheme for computing the *EFV* curve $\lambda(x^k)$ for scenario group k for $k \in \mathcal{K}_a$, $a \in \mathcal{A}^e$, $e = \mathcal{E}, \ldots, 2$ requires us to optimize the set of stochastic subproblems $\{\mathcal{P}^{az} : z \in \mathcal{Z}^{e-1,k}\}$. The expression of the optimal value of the objective function related



$\mathcal{A}^3 = \{19, \ldots, 25, \ldots, 30, \ldots, 36\}$; $\mathcal{G}_8 = \{10, \ldots, 18\}$
$\mathcal{C}_{27} = \{27, 53, 54, 105, \ldots, 108\}$; $\mathcal{A}_8 = \{25, \ldots, 30\}$

Figure 2: Multi-period scenario tree, 3 stages

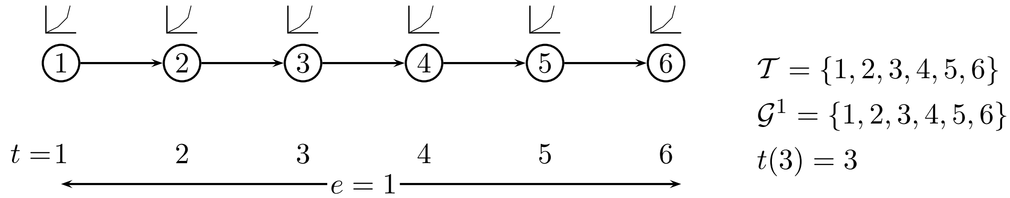


Figure 3: Multi-period scenario tree, stage $e = 1$

to a small perturbation of the value of the linking variables in the given *reference level* gives information for computing the curve.

In effect, the curve $\lambda(x^k)$ can be obtained by the linearization (4.1) of the objective function of subproblem (3.1) around the given value $\overline{x}^{kz}$ of the linking variables, for $z \in \mathcal{Z}^{e-1,k}$ such that $k \in \mathcal{K}_a$.

$$\sigma^{az}(x^k) \approx \hat{\sigma}^{az} + \frac{\partial \sigma^{az}}{\partial x^k}(\bar{x}^{kz})^T(x^k - \overline{x}^{kz}), \tag{4.1}$$

where $\hat{\sigma}^{az}$ is the optimal value of the objective function (3.1), and the vector $\frac{\partial \sigma^{az}}{\partial x^k}(\overline{x}^{kz})$ is the gradient of the objective function of subproblem $\mathcal{P}^{az}$ with respect to the value of the vector x^k of the linking variables related to *reference level* z. Let

$$\frac{\partial \sigma^{az}}{\partial x^k}^T(\bar{x}^{kz}) \equiv \pi^{kaz}, \tag{4.2}$$

where $\pi^{kaz} = \left\{\pi_j^{kaz}\right\}$ and π_j^{kaz} is the scalar that gives the estimation of the objective function increase (that can be negative) due to an increment of the j-th component of the *rhs* of the constraints $x^k = \overline{x}^{kz}$ in subproblem (3.1).

From where the linear approximation of the function (4.1) can be expressed

$$\sigma^{az}(x^k) \approx \mu^{kaz} + \pi^{kaz} x^k, \tag{4.3}$$

where μ^{kaz} is the constant term

$$\mu^{kaz} = \hat{\sigma}^{az} - \pi^{kaz}\overline{x}^{kz}. \tag{4.4}$$

The $\mu-$ and $\pi-$ expected values over the scenario groups will be

$$\mu^{kz} = \sum_{a \in \mathcal{A}_k} w^a \mu^{kaz}, \tag{4.5}$$

$$\pi^{kz} = \sum_{a \in \mathcal{A}_k} w^a \pi^{kaz}. \tag{4.6}$$

Notice that w^a is the weight assigned to the subtree rooted by node a from set A_k.

For each *reference level* $z \in \mathcal{Z}^{e-1,k}$ it results that λ^{kz}, the estimation of λ^k, can be expressed

$$\lambda^{kz} = \mu^{kz} + \pi^{kz} x^k, \tag{4.7}$$

from where it results

$$\lambda^k = \max\{\mu^{kz} + \pi^{kz} x^k \quad \forall z \in \mathcal{Z}^{e-1,k}\}, \qquad \forall k \in \mathcal{K}_a, \tag{4.8}$$

provided that it is a convex function. Otherwise, an adjustment is required to preserve this property.

Finally, the subproblem (3.1) has the following expression for $e = 1$,

$$\begin{array}{lll} \min & \sum_{g \in \mathcal{G}^1} \left(c^g x^g + \lambda^g \right) & \\ s.t. & \sum_{k \in \{g\} \bigcup \mathcal{K}_g} A_{t(g)}^{t(k)} x^k = b^g & \forall g \in \mathcal{G}^1 \\ & x^g \in \mathcal{X} & \forall g \in \mathcal{G}^1 \\ & \lambda^g \geq \mu^{gz} + \pi^{gz} x^g & \forall z \in \mathcal{Z}^{e'g}, g \in \mathcal{G}^1, e' \in \mathcal{E} : e' > 1. \end{array} \tag{4.9}$$

Stopping criteria

For obtaining the solution offered by the approach a *front-to-back* step is performed at each iteration of the algorithm, by considering the EFV curve for each step. The procedure is stopped when a given number of iterations is performed or the solution value of two consecutive iterations is close enough.

5 Production planning under uncertainty

5.1 Problem statement

The problem consists of deciding how much production and, where such is the case, how much product demand loss can be expected at each period along a time horizon. The production capacity constraints, the product stock limitations, some logistic constraints related to the production lot sizing and the product demand requirements should be satisfied at a minimum cost. There is a vast amount of literature on the deterministic version of the problem. See the seminal paper of [25] for considering only continuous variables. See [4, 9, 17, 20, 23, 22, 27], among others, for considering lot sizing limitations and other logical constraints (and, then, considering 0–1 variables).

However, very frequently the production decisions must be made in the presence of uncertainty in several important parameters, such as production cost, product demand and resource availability along a multistage time horizon. We present below two equivalent models for production planning, where the uncertainty is treated via a scenario tree based scheme, such that the occurrence of the events is represented by a multistage scenario tree.

5.2 Complete recourse deaggregate mixed 0–1 *DEM*

The following is the notation for the sets and parameters used in the tactical production planning model.
Sets:

$\mathcal{J}$, set of products.

$\mathcal{R}$, set of resources.

Deterministic parameters:

$\widehat{N}$, maximum number of products to be produced in a single time period.

$\underline{X}_{jt}, \overline{X}_j$, conditional minimum and maximum volume of product j that can be produced at time period t, respectively, if any, for $j \in \mathcal{J}$, $t \in \mathcal{T}$.

$\overline{S}_j$, maximum volume of product j that can be in stock at any time period, for $j \in \mathcal{J}$.

o_{rj}, unit capacity consumption of resource r by product j, for $r \in \mathcal{R}, j \in \mathcal{J}$.

h_j, unit holding cost of product j at any time period, for $j \in \mathcal{J}$.

p_j, unit lost demand penalty for product j, for $j \in \mathcal{J}$.

f_j, fixed cost to be incurred for producing product j at any time period, for $j \in \mathcal{J}$.

Uncertain parameters under scenario group g, for $g \in \mathcal{G}$:

O_r^g, available capacity of resource r at time period $t(g)$, for $r \in \mathcal{R}$.

D_j^g, demand of product j, for $j \in \mathcal{J}$.

c_j^g, unit processing cost of product j, for $j \in \mathcal{J}$.

Variables under scenario group g, for $g \in \mathcal{G}$:

δ_j^g, 0–1 variable such that its value is 1 if product j is produced, and 0 otherwise, for $j \in \mathcal{J}$.

$x_j^{gg'}$, production volume of product j at time period $t(g)$ to satisfy the demand from time period $t(g')$ under scenario group g', for $j \in \mathcal{J}$, $g' \in \mathcal{S}_g$. Notice that the production volume $x_j^{gg'}$ will be in stock during the periods $t(g), t(g)+1, \ldots, t(g')-1$.

y_j^g, lost demand of product j from time period $t(g)$ under scenario group g, for $j \in \mathcal{J}$, $g \in \mathcal{G}$.

The following is a *compact* representation of the *DEM* for the *multistage* stochastic problem with *complete recourse.*

Objective

Determine the production and stock management policy to minimize the expected production and stock cost and the lost demand penalty plus the production fixed cost over the scenarios along the time horizon, subject to the constraints (5.2)- (5.10).

$$\min \sum_{g\in\mathcal{G}} \sum_{j\in\mathcal{J}} \Big[\sum_{g'\in\mathcal{S}_g} w^{g'} \Big(c_j^g + h_j(t(g') - t(g)) \Big) x_j^{gg'} + w^g (p_j y_j^g + f_j \delta_j^g) \Big] \tag{5.1}$$

Constraints

$$\sum_{j\in\mathcal{J}} o_{rj} \sum_{g'\in\mathcal{P}_g^k\cup\{g\}} x_j^{gg'} \le O_r^g \quad \forall r\in\mathcal{R}, k\in\Omega_g, g\in\mathcal{G} \tag{5.2}$$

$$\underline{X}_{j,t(g)}\delta_j^g \le \sum_{g'\in\mathcal{P}_g^k\cup\{g\}} x_j^{gg'} \le \overline{X}_j\delta_j^g \quad \forall j\in\mathcal{J}, k\in\Omega_g, g\in\mathcal{G} \tag{5.3}$$

$$\sum_{j\in\mathcal{J}} \delta_j^g \le \widehat{N} \quad \forall g\in\mathcal{G} \tag{5.4}$$

$$\sum_{\ell\in\mathcal{K}_g\cup\{g\}} x_j^{\ell g} + y_j^g = D_j^g \quad \forall j\in\mathcal{J}, g\in\mathcal{G} \tag{5.5}$$

$$\sum_{\ell\in\mathcal{K}_g\cup\{g\}} \sum_{g'\in\mathcal{P}_g^k} x_j^{\ell g'} \le \overline{S}_j \quad \forall j\in\mathcal{J}, k\in\Omega_g, g\in\mathcal{G} \tag{5.6}$$

$$\sum_{g'\in\mathcal{P}_g^k} x_j^{gg'} = \sum_{g'\in\mathcal{P}_g^{k+1}} x_j^{gg'} \quad \forall j\in\mathcal{J}, k\in\Omega_g, g\in\mathcal{G} \tag{5.7}$$

$$x_j^{gg'} \ge 0 \quad \forall j\in\mathcal{J}, g'\in\mathcal{S}_g, g\in\mathcal{G} \tag{5.8}$$

$$y_j^g \ge 0 \quad \forall j\in\mathcal{J}, g\in\mathcal{G} \tag{5.9}$$

$$\delta_j^g \in \{0,1\} \quad \forall j\in\mathcal{J}, g\in\mathcal{G}. \tag{5.10}$$

The knapsack constraints (5.2) ensure that the consumption of the resources does not exceed the availability. Constraints (5.3) define the semi-continuous character of the production volume. The cover induced constraints (5.4) do not allow to produce more products in a single time period than the maximum allowed. Constraints (5.5) define the demand balance equations, such that the demand deficit is lost. Constraints (5.6) give the upper bounds of the product stock. Constraints (5.7) force the same value for the production volume for a given scenario group and, so, independently of the scenario to occur, thus satisfying the non-anticipativity principle.

The instances of the mixed 0–1 *DEM* (5.1)-(5.10) can have such large dimensions that the using of state-of the-art optimization engines can make it unaffordable. Benders, Lagrangian and Branch-and-Fix Coordination decomposition schemes can be used, although the instances dimensions should be medium-sized.

5.3 Complete recourse aggregate mixed 0–1 *DEM* [3]

Variables:

$x_j^g = \sum_{g' \in \mathcal{P}_g^k \cup \{g\}} x_j^{gg'}$, production volume of product j under scenario group g, for any scenario k in group g.

$s_j^g = \sum_{\ell \in \mathcal{K}_g \cup \{g\}} \sum_{g' \in \mathcal{P}_g^k} x_j^{\ell g'}$, stock volume of product j under scenario group g for any scenario k in group g.

The aggregate model is as follows,

$$\min \sum_{g \in \mathcal{G}} w^g \sum_{j \in \mathcal{J}} \left[c_j^g x_j^g + h_j s_j^g + p_j y_j^g + f_j \delta_j^g \right] \tag{5.11}$$

subject to

$$\sum_{j \in \mathcal{J}} o_{rj} x_j^g \leq O_r^g \quad \forall r \in \mathcal{R}, g \in \mathcal{G} \tag{5.12}$$

$$\underline{X}_{j,t(g)} \delta_j^g \leq x_j^g \leq \overline{X}_j \delta_j^g \quad \forall j \in \mathcal{J}, g \in \mathcal{G} \tag{5.13}$$

$$\sum_{j \in \mathcal{J}} \delta_j^g \leq \widehat{N} \quad \forall g \in \mathcal{G} \tag{5.14}$$

$$s_j^{\rho(g)} + x_j^g = D_j^g + s_j^g - y_j^g \quad \forall j \in \mathcal{J}, g \in \mathcal{G} \tag{5.15}$$

$$0 \leq s_j^g \leq \overline{S}_j \quad \forall j \in \mathcal{J}, g \in \mathcal{G} \tag{5.16}$$

$$y_j^g \geq 0 \quad \forall j \in \mathcal{J}, g \in \mathcal{G} \tag{5.17}$$

$$\delta_j^g \in \{0, 1\} \quad \forall j \in \mathcal{J}, g \in \mathcal{G} \tag{5.18}$$

It is well known that the deterministic version of model (5.1)-(5.10) is tighter than the deterministic version of model (5.11)-(5.18), see [27]. Althogh the results are not shown here, its stochastic version does need more elapsed time and memory than the latter model.

6 Computational experience

We report the computational experience obtained when solving the multistage stochastic mixed 0–1 model for a set of three instance testbeds of the tactical production planning problem. They have been randomly generated. (They are available from the authors on request). The 1st testbed is included by 24 small-scale instances, the 2nd testbed is included by 24 medium-scale instances, and the 3rd testbed is included by the remaining 16 instances, which are very large-scale.

Our algorithm has been implemented in an experimental C code. It uses the optimization engine CPLEX v9.1 for solving the MIP subproblems for each stage. The computational experiments were conducted on a SUN WS W2100z with a 2.2Ghz optaron processor, 4Gb of RAM and the operating system Linux Enterprise 3.

The stopping criterion that we have used for the SDP approach is that the solution does not change in two consecutive *back-to-front* iterations.

The tables 1, 5 and 8 give the problem dimensions, the number of scenario groups and the number of scenarios. The tables 2, 6 and 9 give the dimensions of the *DEM*, compact representation, where the headings are: m, number of constraints; $n01$, number of 0-1 variables; nc, number of continuous variables; nel, number of nonzero elements in the constraint matrix; and $dens$, constraint matrix density (in %). Note that the dimensions of the cases are very high, even the instances of the 1st testbed. We notice in Table 2 that the so-called "small" instances have 50000+ constraints, 14400+ 0–1 variables and 36000 continuous variables in the big cases. The tables 3, 7 and 10 show the main results of our computational experimentation for solving the original problem. The headings are as follows: Z_{LP}, solution value of the LP relaxation of the original problem; Z_{IP}, value of the incumbent solution for the original problem that has been obtained by either using the plain CPLEX or our *SDP* proposal; nn, number of explored BB nodes; T_{IP}, elapsed time (secs.) for obtaining the solution; $niter$, number of the *back-to-front SDP* iterations; Nz, number of reference levels that has been used at each stage; $nprob$, number of MIP subproblems to solve along the stages; GG, goodness gap computed as $(Z_{IP}^{SDP-CPLEX} - Z_{IP}^{CPLEX})/Z_{IP}^{SDP-CPLEX}\%$. Time limit: 8 hours.

We can observe in Table 3 that the solution values of CPLEX and our *SDP* approach are very similar for the 1st testbed, but the total time required by *SDP* is two, three and four orders of magnitude smaller (depending on the instances) than the time required by CPLEX. Note that the *SDP* time goes from 1 second to 1 minute. Three stages are considered for the *SDP* runs. As an example, for the instances with tree structure $1^6 3^2 2^3$, the first stage has 6 periods with one node each, the second stage has the structure 3^2 (there are 2 periods with 3 successors per ancestor node) and the third stage has the structure 2^3 (there are 3 periods with 2 successors per ancestor node), in total 11 time periods, 144 nodes (i.e., scenario groups) and 72 scenarios, see Figure 2. The dimensions of the MIP subproblems are given in Table 4, the headings are as in Table 2. The *SDP* approach cannot win CPLEX (GG is positive) but the solution values are comparable.

Table 6 shows the dimensions of the "medium" scale instances, some of which have 325000+ constraints, 85500 0–1 variables and 200000+ continuous variables. Table 7 shows the computational comparison. Again CPLEX gives better solution values, although GG has been reduced and the time differences are similar to the time differences for the 1st testbed. For instance, SDP requires 5 minutes in case c48 to provide a solution whose GG is 0.8%, while CPLEX has reached the time limit (8 hours) of computing. We can observe that the number of scenarios and scenario groups have grown from $|\Omega| = 72$ and $|\mathcal{G}| = 144$ to 432 and 855, respectively. The dimensions of the MIP subproblems are given in Table 4.

Table 8 shows the dimensions of the 3rd testbed. Notice that we have up to $|\Omega| =$

7776 scenarios, $|\mathcal{G}| = 11684$ scenario groups and a big number of MIP subproblems to solve, see Table 4. This testbed has two characteristics: 5 stages are considered with 14 and 16 time periods, and the dimensions of the instances c61 to c64 are one order of magnitude bigger than the dimensions of the instances in the 1st and 2nd testbeds, resulting in one million+ 0–1 variables, almost three million continuous variables and four million+ constraints, see Table 9. Given the large dimensions of these last instances, CPLEX cannot provide a solution for the original problem within the time limit, it cannot even solve the LP relaxation. However, the SDP approach produces a solution for all instances in very small elapsed time (less than 90 minutes) given the model dimensions, see Table 10.

Table 1: Problem dimensions and number of scenarios. 1st testbed

Case	scenario tree	$\|J\|$	$\|R\|$	$\|T\|$	$\|\mathcal{G}\|$	$\|\Omega\|$
c1	$1^6 2^3 3^2$	10	2	11	116	72
c2	$1^6 2^3 3^2$	10	4	11	116	72
c3	$1^6 3^2 2^3$	10	2	11	144	72
c4	$1^6 2^3 3^2$	20	4	11	116	72
c5	$1^6 2^3 3^2$	20	10	11	116	72
c6	$1^6 3^2 2^3$	20	4	11	144	72
c7	$1^6 3^2 2^3$	20	10	11	144	72
c8	$1^6 2^3 3^2$	30	10	11	116	72
c9	$1^6 3^2 2^3$	30	10	11	144	72
c10	$1^6 2^3 3^2$	40	10	11	116	72
c11	$1^6 2^3 3^2$	40	15	11	116	72
c12	$1^6 2^3 3^2$	50	10	11	116	72
c13	$1^6 2^3 3^2$	50	15	11	116	72
c14	$1^6 3^2 2^3$	40	10	11	144	72
c15	$1^6 2^3 3^2$	50	20	11	116	72
c16	$1^6 3^2 2^3$	40	15	11	144	72
c17	$1^6 3^2 2^3$	50	10	11	144	72
c18	$1^6 3^2 2^3$	50	15	11	144	72
c19	$1^6 3^2 2^3$	50	20	11	144	72
c20	$1^6 2^3 3^2$	100	20	11	116	72
c21	$1^6 2^3 3^2$	100	30	11	116	72
c22	$1^6 3^2 2^3$	100	20	11	144	72
c23	$1^6 3^2 2^3$	100	30	11	144	72
c24	$1^6 3^2 2^3$	100	40	11	144	72

7 Conclusions and future work

A *Stochastic Dynamic Programming* approach has been presented in the paper. The uncertainty is represented by a multistage scenario tree. Linking variables between con-

Table 2: DEM dimensions. Compact Representation. 1st testbed

		Complete DEM					Scenario Model				
case	scenario tree	m	$n01$	nc	nel	dens	m	$n01$	nc	nel	dens
c1	$1^6 2^3 3^2$	4268	1160	2760	11890	0.071	463	110	320	1200	0.602
c2	$1^6 2^3 3^2$	4500	1160	2760	13978	0.079	485	110	320	1420	0.680
c3	$1^6 3^2 2^3$	5472	1440	3600	15110	0.054	463	110	320	1200	0.602
c4	$1^6 2^3 3^2$	8420	2320	5520	27492	0.041	915	220	640	2840	0.360
c5	$1^6 2^3 3^2$	9116	2320	5520	39440	0.055	981	220	640	4160	0.493
c6	$1^6 3^2 2^3$	10800	2880	7200	35980	0.033	915	220	640	2840	0.360
c7	$1^6 3^2 2^3$	11664	2880	7200	48652	0.041	981	220	640	4160	0.493
c8	$1^6 2^3 3^2$	13036	3480	8280	59566	0.038	1411	330	960	6240	0.342
c9	$1^6 3^2 2^3$	16704	4320	10800	74418	0.029	1411	330	960	6240	0.342
c10	$1^6 2^3 3^2$	16956	4640	11040	77604	0.029	1841	440	1280	8320	0.262
c11	$1^6 2^3 3^2$	17536	4640	11040	96396	0.035	1896	440	1280	10520	0.322
c12	$1^6 2^3 3^2$	20876	5800	13800	97962	0.023	2271	550	1600	10400	0.212
c13	$1^6 2^3 3^2$	21456	5800	13800	119538	0.028	2326	550	1600	13150	0.262
c14	$1^6 3^2 2^3$	21744	5760	14400	97880	0.022	1841	440	1280	8320	0.262
c15	$1^6 2^3 3^2$	22036	5800	13800	145754	0.033	2381	550	1600	15900	0.310
c16	$1^6 3^2 2^3$	22464	5760	14400	119768	0.026	1896	440	1280	10520	0.322
c17	$1^6 3^2 2^3$	26784	7200	18000	123790	0.018	2271	550	1600	10400	0.212
c18	$1^6 3^2 2^3$	27504	7200	18000	151582	0.021	2326	550	1600	13150	0.262
c19	$1^6 3^2 2^3$	28224	7200	18000	179806	0.025	2381	550	1600	15900	0.310
c20	$1^6 2^3 3^2$	41636	11600	27600	283504	0.017	4531	1100	3200	31800	0.163
c21	$1^6 2^3 3^2$	42796	11600	27600	378972	0.022	4641	1100	3200	42800	0.214
c22	$1^6 3^2 2^3$	53424	14400	36000	361340	0.013	4531	1100	3200	31800	0.163
c23	$1^6 3^2 2^3$	54864	14400	36000	478988	0.017	4641	1100	3200	42800	0.214
c24	$1^6 3^2 2^3$	56304	14400	36000	593180	0.020	4751	1100	3200	53800	0.263

secutive and non-consecutive time periods are allowed. The proposed scheme utilizes the scenario tree in a *back-to-front* way. A gradient based perturbation of the objective function around a set of *reference levels* gives the estimation of the coefficients of the *EFV* curves for the expected impact of the variables in the objective function of the original problem. For assessing its validity a production planning problem with logical constraints is considered as a pilot case in both versions, namely, aggregate and deaggregate models. A (small) stochastic multistage mixed 0–1 problem is solved for each *reference level* from a given set and each starting subtree associated with the stages in the scenario tree. The first conclusion that can be drawn from the computational experience with the pilot problem is that the results (not shown in the paper) for the deaggregate model are not good given the high dimensions that are required. It is well known that this has a better performance than the aggregate model in the deterministic setting, but it is of no use in the stochastic setting. However, the modelization technique that we have used is new as far we know and the approach can be useful for other problems. Additionally, the computational results for the aggregate model are comparable with the results obtained by plain use of CPLEX, a state-of-the-art optimization engine, for "small" and "medium"-scale instances, but it gives results for large-scale instances (over one million 0–1 variables for 7000+ scenarios, 11000+ scenario groups, 5 stages and 16 time periods) where CPLEX cannot find any solution (even for the LP relaxation) within the time limit (8 hours). In any case, the proposed SDP approach needs very little computing time (less than 90 minutes for the biggest instances).

As a future work we are planning to study the effect of the cardinality of the sets of *reference levels* in the objective function of the original problem. Another task which still remains is the parallelization of the optimization of the models attached to the subtrees with the roots as the starting scenario groups at each stage of the scenario tree and each *reference level*. We anticipate that the elapsed time could be drastically reduced.

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Table 3: Solution values. 1st testbed

			Plain CPLEX				$SDP-CPLEX$					
case	scenario tree	Z_{LP}	t_{LP}	Z_{IP}	t_{IP}	nn	Z_{IP}	t_{IP}	$niter$	Nz	$n$prob	$GG\%$
c1	$1^6 2^3 3^2$	15813	0.01	830310	0.04	25	832741	0.32	2	9	714	0.2
c2	$1^6 2^3 3^2$	15076	0.02	358322	0.16	76	361996	0.56	2	9	714	1.0
c3	$1^6 3^2 2^3$	13956	0.02	916801	0.06	2	920134	0.59	2	9	812	0.3
c4	$1^6 2^3 3^2$	34611	0.05	1293765	0.15	39	1331129	2.71	4	17	2142	2.8
c5	$1^6 2^3 3^2$	29393	0.05	1455129	0.11	45	1490031	0.9	2	9	714	2.3
c6	$1^6 3^2 2^3$	33705	0.04	1106305	61.12	43611	1127835	1.4	2	9	812	1.9
c7	$1^6 3^2 2^3$	26116	0.06	946191	2.05	807	956362	1.62	2	9	812	1.0
c8	$1^6 2^3 3^2$	42631	0.07	1557732	0.68	256	1588201	1.72	2	9	714	1.9
c9	$1^6 3^2 2^3$	46125	0.09	1238900	10.98	4173	1246994	2.34	2	9	812	0.6
c10	$1^6 2^3 3^2$	60061	0.07	2515217	1.34	608	2576060	2.06	2	9	714	2.3
c11	$1^6 2^3 3^2$	63256	0.10	2161413	2.06	820	2182869	6.72	4	17	2142	0.9
c12	$1^6 2^3 3^2$	78626	0.12	2930478	2.18	823	2951628	4.88	3	13	1326	0.7
c13	$1^6 2^3 3^2$	79199	0.14	2977623	9.83	4449	3000054	10.51	4	17	2142	0.7
c14	$1^6 3^2 2^3$	56885	0.12	2268162	5.03	1276	2281700	2.69	2	9	812	0.6
c15	$1^6 2^3 3^2$	74120	0.16	2649567	4.99	1570	2675629	3.46	2	9	714	0.9
c16	$1^6 3^2 2^3$	60444	0.13	2641603	0.85	97	2665565	8.61	4	17	2436	0.9
c17	$1^6 3^2 2^3$	69941	0.17	2176069	19861.96	3757360	2214307	3.81	2	9	812	1.7
c18	$1^6 3^2 2^3$	72559	0.18	3076014	737.77	204979	3100176	3.65	2	9	812	0.7
c19	$1^6 3^2 2^3$	70055	0.19	2765041 [1]	28800.01	6627785	2817617	27.1	6	25	4988	1.8
c20	$1^6 2^3 3^2$	163727	0.31	6689224	6.51	1007	6736526	6.48	2	9	714	0.7
c21	$1^6 2^3 3^2$	156125	0.33	4488014	244.71	57044	4597174	26.96	4	17	2142	2.3
c22	$1^6 3^2 2^3$	160219	0.44	5982664	10377.22	1304416	6033771	29.37	4	17	2436	0.8
c23	$1^6 3^2 2^3$	162530	0.47	5635912 [1]	28800.03	4412541	5684449	71.79	6	25	4988	0.8
c24	$1^6 3^2 2^3$	158855	0.54	5227253 [1]	28800.01	4248981	5279983	59.37	5	21	3596	0.9

[1] Time limit exceed maximum 8 hours.

[2] Stop. Out of memory.

Table 4: MIP subproblems. Dimensions

	smallest subproblem			greatest subproblem		
bed	m	$n01$	nc	m	$n01$	nc
testbed 1	62	40	100	1287	700	1800
testbed 2	258	60	180	5765	1500	3800
testbed 3	139	30	100	4803	1300	3100

Note: The smallest and largest models are given by the smallest and greatest number of 0-1 variables.

Table 5: Problem dimensions and number of scenarios. 2nd testbed

Case	scenario tree	$\|J\|$	$\|R\|$	$\|T\|$	$\|\mathcal{G}\|$	$\|\Omega\|$
c25	$1^6 2^4 3^3$	10	2	13	660	432
c26	$1^6 2^4 3^3$	10	4	13	660	432
c27	$1^6 3^3 2^4$	10	2	13	855	432
c28	$1^6 3^3 2^4$	10	4	13	855	432
c29	$1^6 2^4 3^3$	20	4	13	660	432
c30	$1^6 2^4 3^3$	20	10	13	660	432
c31	$1^6 3^3 2^4$	20	4	13	855	432
c32	$1^6 3^3 2^4$	20	10	13	855	432
c33	$1^6 2^4 3^3$	30	10	13	660	432
c34	$1^6 2^4 3^3$	40	10	13	660	432
c35	$1^6 2^4 3^3$	40	15	13	660	432
c36	$1^6 3^3 2^4$	30	10	13	855	432
c37	$1^6 2^4 3^3$	50	10	13	660	432
c38	$1^6 2^4 3^3$	50	15	13	660	432
c39	$1^6 2^4 3^3$	50	20	13	660	432
c40	$1^6 3^3 2^4$	40	10	13	855	432
c41	$1^6 3^3 2^4$	40	15	13	855	432
c42	$1^6 3^3 2^4$	50	10	13	855	432
c43	$1^6 3^3 2^4$	50	15	13	855	432
c44	$1^6 3^3 2^4$	50	20	13	855	432
c45	$1^6 2^4 3^3$	100	20	13	660	432
c46	$1^6 2^4 3^3$	100	30	13	660	432
c47	$1^6 3^3 2^4$	100	20	13	855	432
c48	$1^6 3^3 2^4$	100	30	13	855	432

Table 6: DEM dimensions. Compact Representation. 2nd testbed

		Complete DEM					Scenario Model				
case	scenario tree	m	$n01$	nc	nel	dens	m	$n01$	nc	nel	dens
c25	$1^6 2^4 3^3$	24060	6600	15480	66590	0.013	549	130	380	1420	0.507
c26	$1^6 2^4 3^3$	25380	6600	15480	77810	0.014	575	130	380	1680	0.573
c27	$1^6 3^3 2^4$	32445	8550	21330	90530	0.009	549	130	380	1420	0.507
c28	$1^6 3^3 2^4$	34155	8550	21330	104210	0.010	575	130	380	1680	0.573
c29	$1^6 2^4 3^3$	47460	13200	30960	159580	0.008	1085	260	760	3360	0.304
c30	$1^6 2^4 3^3$	51420	13200	30960	226240	0.010	1163	260	760	4920	0.415
c31	$1^6 3^3 2^4$	64035	17100	42660	209275	0.005	1085	260	760	3360	0.304
c32	$1^6 3^3 2^4$	69165	17100	42660	299050	0.007	1163	260	760	4920	0.415
c33	$1^6 2^4 3^3$	73500	19800	46440	333750	0.007	1673	390	1140	7380	0.288
c34	$1^6 2^4 3^3$	95580	26400	61920	447860	0.005	2183	520	1520	9840	0.221
c35	$1^6 2^4 3^3$	98880	26400	61920	552800	0.006	2248	520	1520	12440	0.271
c36	$1^6 3^3 2^4$	99045	25650	63990	430620	0.005	1673	390	1140	7380	0.288
c37	$1^6 2^4 3^3$	117660	33000	77400	555370	0.004	2693	650	1900	12300	0.179
c38	$1^6 2^4 3^3$	120960	33000	77400	684730	0.005	2758	650	1900	15550	0.221
c39	$1^6 2^4 3^3$	124260	33000	77400	836530	0.006	2823	650	1900	18800	0.261
c40	$1^6 3^3 2^4$	128925	34200	85320	593825	0.004	2183	520	1520	9840	0.221
c41	$1^6 3^3 2^4$	133200	34200	85320	717800	0.005	2248	520	1520	12440	0.271
c42	$1^6 3^3 2^4$	158805	42750	106650	712570	0.003	2693	650	1900	12300	0.179
c43	$1^6 3^3 2^4$	163080	42750	106650	892975	0.004	2758	650	1900	15550	0.221
c44	$1^6 3^3 2^4$	167355	42750	106650	1072525	0.004	2823	650	1900	18800	0.261
c45	$1^6 2^4 3^3$	234660	66000	154800	1635440	0.003	5373	1300	3800	37600	0.137
c46	$1^6 2^4 3^3$	241260	66000	154800	2142320	0.004	5503	1300	3800	50600	0.180
c47	$1^6 3^3 2^4$	316755	85500	213300	2134790	0.002	5373	1300	3800	37600	0.137
c48	$1^6 3^3 2^4$	325305	85500	213300	2798270	0.003	5503	1300	3800	50600	0.180

Table 7: Solution values. 2nd testbed

			Plain CPLEX					$SDP-CPLEX$					
case	scenario tree	Z_{LP}	t_{LP}	Z_{IP}	t_{IP}	nn	Z_{IP}	t_{IP}	$niter$	Nz	$n$prob	$GG\%$
c25	$1^6 2^4 3^3$	15798	0.09	1075621	0.23	20	1085157	1.98	2	9	714	0.9
c26	$1^6 2^4 3^3$	15696	0.12	1146946	0.14	5	1149136	3.81	3	13	1377	0.2
c27	$1^6 3^3 2^4$	17321	0.18	717161 [1]	28800.03	4772630	728435	4.70	2	9	812	1.5
c28	$1^6 3^3 2^4$	15627	0.15	927135	1.56	153	932886	3.40	2	9	812	0.6
c29	$1^6 2^4 3^3$	35511	0.24	1683262	0.64	18	1708830	4.60	2	9	714	1.5
c30	$1^6 2^4 3^3$	26434	0.30	1745853	1.11	90	1752770	5.75	2	9	714	0.4
c31	$1^6 3^3 2^4$	39534	0.37	1907302	985.94	184289	1921383	8.63	2	9	812	0.7
c32	$1^6 3^3 2^4$	36108	0.40	1092750 [1]	28800.02	3276916	1106512	61.85	5	21	3770	1.2
c33	$1^6 2^4 3^3$	45889	0.49	1669375	429.84	70739	1686014	18.72	3	13	1377	1.0
c34	$1^6 2^4 3^3$	64747	0.62	2298606	1989.33	244222	2367019	43.37	4	17	2244	2.9
c35	$1^6 2^4 3^3$	58802	0.69	2331871 [1]	28800.02	3564053	2354928	15.29	2	9	714	1.0
c36	$1^6 3^3 2^4$	50731	0.67	1721782 [1]	28800.02	3059853	1739389	18.29	2	9	812	1.0
c37	$1^6 2^4 3^3$	85195	0.80	3577998 [1]	28800.01	3845492	3614010	131.9	6	25	4590	1.0
c38	$1^6 2^4 3^3$	84556	0.93	3277473 [1]	28800.04	3604127	3295960	19.88	2	9	714	0.6
c39	$1^6 2^4 3^3$	83147	1.00	3612532 [1]	28800.02	3770371	3641829	200.95	7	29	6069	0.8
c40	$1^6 3^3 2^4$	62266	0.92	3264867 [1]	28800.02	2967111	3289642	20.68	2	9	812	0.8
c41	$1^6 3^3 2^4$	73444	0.97	3387406 [1]	28800.02	2799542	3407129	45.86	3	13	1566	0.6
c42	$1^6 3^3 2^4$	92696	1.12	4329560 [1]	28800.01	2424996	4362136	22.35	2	9	812	0.7
c43	$1^6 3^3 2^4$	93258	1.35	3510296 [1]	28800.02	1989219	3539830	70.07	3	13	1566	0.8
c44	$1^6 3^3 2^4$	87761	1.44	4214406 [1]	28800.01	2442284	4259200	148.27	5	21	3770	1.1
c45	$1^6 2^4 3^3$	196912	2.20	8037153 [1]	28800.02	2347277	8125403	39.75	2	9	714	1.1
c46	$1^6 2^4 3^3$	187742	2.57	8089552 [1]	28800.01	2329950	8143174	109.22	3	13	1377	0.7
c47	$1^6 3^3 2^4$	192924	2.51	7160902 [2]	13435.00	470521	7225673	245.84	4	17	2552	0.9
c48	$1^6 3^3 2^4$	176461	3.22	6607400 [2]	13855.33	439349	6663366	302.48	4	17	2552	0.8

[1] Time limit (8 hours) exceeded.

[2] Stop. Out of memory.

Table 8: Problem dimensions and number of scenarios. 3rd testbed

Case	scenario tree	$\|J\|$	$\|R\|$	$\|T\|$	$\|\mathcal{G}\|$	$\|\Omega\|$
c49	$1^6 2^2 2^2 3^2 3^2$	10	2	14	1956	1296
c50	$1^6 2^2 2^2 3^2 3^2$	10	4	14	1956	1296
c51	$1^6 3^2 3^2 2^2 2^2$	10	2	14	2556	1296
c52	$1^6 3^2 3^2 2^2 2^2$	10	4	14	2556	1296
c53	$1^6 3^2 3^2 2^3 2^3$	10	2	16	10332	5184
c54	$1^6 3^2 3^2 2^3 2^3$	10	4	16	10332	5184
c55	$1^6 2^2 2^3 3^2 3^3$	10	4	16	11684	7776
c56	$1^6 2^2 2^3 3^2 3^3$	10	2	16	11684	7776
c57	$1^6 2^2 2^2 3^2 3^2$	100	20	14	1956	1296
c58	$1^6 2^2 2^2 3^2 3^2$	100	40	14	1956	1296
c59	$1^6 3^2 3^2 2^2 2^2$	100	20	14	2556	1296
c60	$1^6 3^2 3^2 2^2 2^2$	100	40	14	2556	1296
c61	$1^6 3^2 3^2 2^3 2^3$	100	20	16	10332	5184
c62	$1^6 3^2 3^2 2^3 2^3$	100	30	16	10332	5184
c63	$1^6 2^2 2^3 3^2 3^3$	100	20	16	11684	7776
c64	$1^6 2^2 2^3 3^2 3^3$	100	30	16	11684	7776

Table 9: DEM dimensions. Compact Representation. 3rd testbed

		Complete DEM					Scenario Model				
case	scenario tree	m	$n01$	nc	nel	dens	m	$n01$	nc	nel	dens
c49	$1^6 2^2 2^2 3^2 3^2$	71148	19560	45720	200966	0.004	592	140	410	1530	0.46
c50	$1^6 2^2 2^2 3^2 3^2$	75060	19560	45720	226394	0.004	620	140	410	1810	0.53
c51	$1^6 3^2 3^2 2^2 2^2$	96948	25560	63720	262898	0.003	592	140	410	1530	0.46
c52	$1^6 3^2 3^2 2^2 2^2$	102060	25560	63720	311462	0.003	620	140	410	1810	0.53
c53	$1^6 3^2 3^2 2^3 2^3$	392436	103320	258120	1115486	0.0007	678	160	470	1750	0.40
c54	$1^6 3^2 3^2 2^3 2^3$	413100	103320	258120	1239470	0.0008	710	160	470	2070	0.46
c55	$1^6 2^2 2^3 3^2 3^3$	448020	116840	272760	1340022	0.0007	710	160	470	2070	0.46
c56	$1^6 2^2 2^3 3^2 3^3$	424652	116840	272760	1199814	0.0007	678	160	470	1750	0.40
c57	$1^6 2^2 2^2 3^2 3^2$	693876	195600	457200	4775444	0.001	5794	1400	4100	40500	0.12
c58	$1^6 2^2 2^2 3^2 3^2$	732996	195600	457200	7903088	0.001	6074	1400	4100	68500	0.20
c59	$1^6 3^2 3^2 2^2 2^2$	946476	255600	637200	6411860	0.0007	5794	1400	4100	40500	0.12
c60	$1^6 3^2 3^2 2^2 2^2$	997596	255600	637200	10555136	0.001	6074	1400	4100	68500	0.20
c61	$1^6 3^2 3^2 2^3 2^3$	3831372	1033200	2581200	26001944	0.0001	6636	1600	4700	46300	0.11
c62	$1^6 3^2 3^2 2^3 2^3$	3934692	1033200	2581200	34277876	0.0002	6796	1600	4700	62300	0.14
c63	$1^6 2^2 2^3 3^2 3^3$	4141364	1168400	2727600	28507632	0.0001	6636	1600	4700	46300	0.11
c64	$1^6 2^2 2^3 3^2 3^3$	4258204	1168400	2727600	38497452	0.0002	6796	1600	4700	62300	0.14

Table 10: Solution values. 3rd testbed

			Plain CPLEX				$SDP-CPLEX$					
case	scenario tree	Z_{LP}	t_{LP}	Z_{IP}	t_{IP}	nn	Z_{IP}	t_{IP}	$niter$	Nz	$n$prob	$GG\%$
c49	$1^6 2^2 2^2 3^2 3^2$	14094	0.39	993334	0.40	0	1023824	6.28	2	9	6874	2.9
c50	$1^6 2^2 2^2 3^2 3^2$	16666	0.38	1005119	1.03	108	1011070	6.02	2	9	6874	0.5
c51	$1^6 3^2 3^2 2^2 2^2$	18402	0.54	772576 [1]	28800.01	2966033	794133	13.25	2	9	11774	2.7
c52	$1^6 3^2 3^2 2^2 2^2$	20511	0.65	863055 [1]	28800.16	3442001	877109	212.02	10	41	195112	1.6
c53	$1^6 3^2 3^2 2^3 2^3$	21403	2.43	772083 [2]	22634.76	433302	706475	53.22	3	13	38012	-9.2
c54	$1^6 3^2 3^2 2^3 2^3$	19224	2.43	673221 [2]	20046.46	465114	706475	49.87	2	9	20468	4.7
c55	$1^6 2^2 2^3 3^2 3^3$	23775	2.52	1126347 [1]	28800.02	2016119	1132367	39.15	2	9	13594	0.5
c56	$1^6 2^2 2^3 3^2 3^3$	21940	2.48	1164688 [2]	26413.66	914895	1132367	22.82	2	9	13594	-2.8
c57	$1^6 2^2 2^2 3^2 3^2$	190641	7.08	7181492 [2]	23378.18	446069	7426324	3952.24	15	61	242063	3.2
c58	$1^6 2^2 2^2 3^2 3^2$	179749	9.29	8758639 [2]	20696.63	557818	8970959	1148.81	7	29	55974	2.3
c59	$1^6 3^2 3^2 2^2 2^2$	183144	10.33	8221657 [2]	5730.48	62720	8448332	920.29	6	25	72326	2.6
c60	$1^6 3^2 3^2 2^2 2^2$	186600	13.98	8605080 [2]	4159.14	49162	8806620	547.56	4	17	35322	2.2
c61	$1^6 3^2 3^2 2^3 2^3$	-	-	-	-	-	9664379	600.03	2	9	20468	-
c62	$1^6 3^2 3^2 2^3 2^3$	-	-	-	-	-	8910226 [3]	3599.53	5	21	90644	-
c63	$1^6 2^2 2^3 3^2 3^3$	-	-	-	-	-	8192060	1815.10	4	17	40782	-
c64	$1^6 2^2 2^3 3^2 3^3$	-	-	-	-	-	7696755 [3]	4477.67	5	21	60202	-

[1] Time limit (8 hours) exceeded.

[2] Stop. Out of memory.

[3] Forced to 5 *back-to-front* iterations.