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TRABAJOS I+D

Stability of systems of linear equations and inequalities: Distance to ill-posedness and metric regularity*

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Abstract

In this paper we consider the parameter space of all the linear systems, in the n-dimensional Euclidean space, with arbitrarily many (possibly infinite) inequalities and a finite amount of equations. This parameter space is endowed with the topology of the uniform convergence of the coefficient vectors by means of an extended distance. Our focus is on the stability of the nominal system in terms of whether or not proximal systems preserve consistency/inconsistency. We pay special attention to the different frameworks coming from either splitting each equation into two inequalities, or treating equations *as* equations. The notable differences arising in the latter setting with respect to the former are emphasized in the paper. Ill-posedness is identified with the boundary of the set of consistent systems and a formula for the distance to ill-posedness is obtained. This formula is applied to derive the modulus of metric regularity of a set-valued mapping describing homogeneous systems.

Key words. Stability, well-posedness, linear systems, distance to ill-posedness, metric regularity.

AMS Subject Classification. 65F22, 90C34, 90C05, 15A39, 49J53.

1 Introduction

In this paper we are concerned with the stability of the linear system (with inequality and equality constraints), in $\mathbb{R}^n$,

$$\sigma := \{a_t' x \geq b_t,\ t \in T;\ a_s' x = b_s,\ s \in S\}, \tag{1}$$

where $T \cap S = \varnothing$, T is an arbitrary *index set*, the functions $t \mapsto a_t \in \mathbb{R}^n$ and $t \mapsto b_t \in \mathbb{R}$ are also arbitrary, S is a finite non-empty set whose cardinal is $m \leq n$, and $a_s \in \mathbb{R}^n$, $b_s \in \mathbb{R}$, for $s \in S$. Here the vectors in $\mathbb{R}^n$ are regarded as

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column-vectors and y' denotes the transpose of $y \in \mathbb{R}^n$. When T is infinite, σ is a linear semi-infinite system.

There exist several papers (see, e.g., [3], [4], [11], [13] and [14]) dealing with the stability and well/ill-posedness of linear inequality systems in this semi-infinite context (T arbitrary), but including no equality constraints in the model (i.e., $S = \varnothing$ in (1)). Although it is well-known that any equation may be split into two inequalities, we will show in this paper that the stability theory in both settings (equations and inequalities, or inequalities only) presents substantial differences. To start with, it is immediate that in the context of inequalities only, the two ones, $a'x \geq b$ and $-a'x \geq -b$, coming from splitting $a'x = b$ may be perturbed as $a'x \geq b+\varepsilon$ and $-a'x \geq -b+\varepsilon$, with $\varepsilon > 0$, yielding inconsistency (infeasibility). On the other hand, it is obvious that small perturbations of the equation $a'x = b$, with $a \neq 0_n$ (the zero vector of $\mathbb{R}^n$), give rise to new consistent equations.

The papers referred above approach the stability of linear inequality systems ($S = \varnothing$) through continuity properties of the feasible set mapping, among other stability criteria. The paper [3] also tackles the stability of the linear semi-infinite optimization problem constrained by such an inequality system. An immediate antecedent to these papers may be found in [1] and [8], which deal with the *continuous case*, where T is a compact Hausdorff space and a_t and b_t depend continuously on $t \in T$. The general case (T, $a(\cdot)$ and $b(\cdot)$ arbitrary) may give rise to certain pathologies with respect to stability which do not occur in the continuous case, as we will show later on in relation to subsets Θ_∞ and Ω_∞ (see Section 2). In [10] the authors analyze the effect that certain specific perturbations provoke on the optimal value of linear semi-infinite programming problems constrained by systems of linear equations and inequalities. Specifically, the authors consider perturbations of either the coefficients of the objective function, or the right hand side of the constraints. In both cases the left hand side of the constraint system remains fixed.

There are, spread out in the literature, many contributions to the stability theory of the feasible set for a class of semi-infinite systems structurally richer than our linear inequality systems (T arbitrary). This class is formed by those systems whose index set T is a compact set in the Euclidean space, defined as solution set of finitely many analytic constraints, and the coefficient functions $a(\cdot)$ and $b(\cdot)$ are assumed to belong to $\mathcal{C}^1(T)$. Obviously, this class of $\mathcal{C}^1$-systems is a subclass of continuous systems. Under suitable hypotheses, [18] (see also [17]) characterizes the topological stability of the feasible set in terms of the Mangasarian-Fromovitz constraint qualification (MFCQ, for short). In this semi-infinite context (with $\mathcal{C}^1$ data), the equivalence between the MFCQ and the metric regularity of the constraints has been established in [15]. More references in relation to metric regularity are given in Section 5.

Other contributions to the stability and well/ill-posedness for linearly constrained systems can be found in the context of conic linear systems (see, e.g., [9], [22] and [23], among others). This context includes our constraint system (1) when T is finite. Unfortunately, when T is infinite and arbitrary the tools developed in [9] and [23] do not apply in our context, in which the parameter space is not a normed space.

In the following paragraphs we go deeply in the description of our model. Associated to σ, we consider the linear inequality system, in $\mathbb{R}^n$:

$$\theta_\sigma := \{a_t'x \geq b_t,\ t \in T;\ a_s'x \geq b_s,\ -a_s'x \geq -b_s,\ s \in S\}. \tag{2}$$

The *parameter space* of all the linear systems in the form (1) will be denoted by Ω, while Θ will represent the set of linear inequality systems indexed as θ_σ; i.e., Θ coincides with the set of systems in the form:

$$\theta := \{a_t'x \geq b_t,\ t \in T;\ a_{s1}'x \geq b_{s1},\ a_{s2}'x \geq b_{s2},\ s \in S\}. \tag{3}$$

In this way, there exists a natural bijection (a isometry, indeed) between Ω and the subset of Θ formed by all the systems in the form (2). Note that Θ may be identified with $(\mathbb{R}^n \times \mathbb{R})^{\overline{T}}$, where

$$\overline{T} := T \cup (S \times \{1, 2\}), \tag{4}$$

and for the sake of brevity the system θ in (3) will be alternatively written in the form $\left\{a_t'x \geq b_t,\ t \in \overline{T}\right\}$. The subset of Ω (respectively, Θ) formed by all the *consistent systems* will be denoted by Ω_c (respectively, Θ_c), while Ω_i and Θ_i represent the corresponding subsets of Ω and Θ formed by all the *inconsistent systems*. According to [13], we distinguish two subsets in Θ_i, which constitute a partition of it: Θ_s, the set of *strongly inconsistent systems* (i.e., those systems having an inconsistent finite subsystem), and $\Theta_w := \Theta_i \backslash \Theta_s$, the set of *weakly inconsistent systems*. Ω_s and Ω_w are defined in analogous way. When different systems are considered in Ω (respectively, in Θ), they and their associated elements will be distinguished by means of sub(super)scripts or by means of accents. So, for example, if σ_1 and $\widetilde{\sigma}$ also belong to Ω, we write $\sigma_1 := \{\left(a_t^1\right)' x \geq b_t^1,\ t \in T;\ \left(a_s^1\right)' x = b_s^1,\ s \in S\}$ and $\widetilde{\sigma} := \{\widetilde{a}_t'x \geq \widetilde{b}_t,\ t \in T;$ $\widetilde{a}_s'x = \widetilde{b}_s,\ s \in S\}$.

We consider Ω and Θ endowed with the topology of the uniform convergence of the coefficient vectors, via the respective *extended distances* $d_\Omega : \Omega \times \Omega \to [0, +\infty]$ and $d_\Theta : \Theta \times \Theta \to [0, +\infty]$ given by:

$$d_\Omega(\sigma_1, \sigma) := \sup_{t \in T \cup S} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|;\ d_\Theta(\theta_1, \theta) := \sup_{t \in \overline{T}} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|,$$

where $\|\cdot\|$ is any given norm in $\mathbb{R}^{n+1}$. When it is clear from the context, we will denote both, d_Ω and d_Θ, by d. Observe that the topology on Ω and Θ does not depend on the norm $\|\cdot\|$ under consideration.

Given $\sigma \in \Omega$ and $\widetilde{\Omega} \subset \Omega$, we will write:

$$d(\sigma, \widetilde{\Omega}) := \inf\left\{d(\sigma, \tilde{\sigma}),\ \tilde{\sigma} \in \widetilde{\Omega}\right\} \in [0, +\infty]$$

where, as usual, $d(\sigma, \varnothing) = +\infty$. The analogous definition can be given for $d(\theta, \widetilde{\Theta})$, with $\theta \in \Theta$ and $\widetilde{\Theta} \subset \Theta$.

At this moment, we advance the following notation: if X is a subset of any topological space, $int(X)$, $cl(X)$, $ext\,(X)$ $(:= X \diagdown cl(X))$ and $bd(X)$ will denote the *interior*, the *closure*, the *exterior*, and the *boundary* of X, respectively.

The stability theory for linear inequality systems as $\{a_t' x \geq b_t,\ t \in T\}$, with T arbitrary, is partially gathered in [12, Chapter 6] (see also references above). Theorem 6.1 in [12] (see also Theorem 3.1 in [13] and Theorem 3.1 in [11]) establishes in this setting ($S = \varnothing$) that condition '$\theta \in int\,(\Theta_c)$' is equivalent to different stability criteria spread out in the literature (see, e.g., [18], [24] and [26]). The paper [5] tackles the stability of systems in Θ from a quantitative point of view. It measures the distance from any given system $\theta \in \Theta$ to $bd\,(\Theta_c)$; so, for instance, starting from $\theta \in int\,(\Theta_c)$, the supremum of the radii of the balls centered at θ and contained in Θ_c is calculated. The referred distance is called there *distance to ill-posedness*, following the terminology introduced by Renegar in [22] (see § 1 in [9] for additional comments).

Now we refer to the structure of the paper. In Section 2 we provide some notation and preliminary results, taken from the context of linear inequalities and adapted to (1). Section 3 approaches the stability of systems (1) with respect to consistency/inconsistency, providing interiority and boundary characterizations, which are synthesized in Theorem 4 and proved along the section. In Section 3 we also investigate to what extent stability in Ω is induced by stability in Θ via the strategy of splitting equations into inequalities. This is done in §3.1. In §3.2 we analyze the specifics of systems of equations and inequalities which require *ad hoc* techniques. Roughly speaking, we can say that well-posedness for inconsistent systems in Ω and in Θ is quite similar; but this is no longer the case for consistent systems, where the role played in Θ by the convex sets C and H (defined in (6)) is now developed by the (in general) non-convex sets E and G (see (8) and (9)). In section 4 we provide a formula for the distance to ill-posedness (distance to consistency/inconsistency). This formula, with clear geometrical features, expresses the referred distance, in the infinite-dimensional (if T is infinite) space Ω, in terms of the distance from the origin to $bd\,(G) \subset \mathbb{R}^{n+1}$. At the end of Section 4 (in §4.1) we apply Theorems 1 and 2 in [9] to derive, in the case in which T is finite and σ is consistent, the distance to ill-posedness as the optimal value of a mathematical program. Finally, Section 5 applies Theorem 9 (in Section 4) to determine the radius of metric regularity of a set-valued mapping describing homogeneous linear inequality and equality systems. In this way we extend to systems of linear equations and inequalities the corresponding result for inequality systems given in [2, Section 3].

2 Preliminaries

In this section we collect the necessary notation, definitions and results that will be used later on. Given $\varnothing \neq X \subset \mathbb{R}^k$, by $conv(X)$, $cone(X)$, and $span(X)$, we denote the *convex hull* of X, the *conical convex hull* of X, and the *linear hull* of X, respectively. It is assumed that $cone\,(X)$ always contains the zero-vector, 0_k, and so $cone(\varnothing) = \{0_k\}$.

The *open unit ball* for the norm $\|\cdot\|$ is represented by B. The *dual norm* of $\|\cdot\|$ is denoted by $\|\cdot\|_*$; i.e., for $u \in \mathbb{R}^k$,

$$\|u\|_* := \max\{u'z \mid \|z\| \leq 1\}.$$

Sequences are usually indexed by $r \in \mathbb{N}$, and $\lim_r$ should be interpreted as $\lim_{r\to+\infty}$.

[12, Theorem 4.4] characterizes, in the context of inequality systems (3), those systems in Θ_c, Θ_w and Θ_s in terms of the so called *second moment cone* and *characteristic cone* associated to θ. As a counterpart the following convex cones, N and K, associated with the system (1) allow us to characterize, in Theorem 1, the systems belonging to Ω_c, Ω_w and Ω_s :

$$\begin{aligned} N &:= cone\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T\right\}\right) + span\left(\left\{\begin{pmatrix} a_s \\ b_s \end{pmatrix},\ s \in S\right\}\right) \\ &= cone\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T;\ \begin{pmatrix} a_s \\ b_s \end{pmatrix},\ s \in S;\ \begin{pmatrix} -a_s \\ -b_s \end{pmatrix},\ s \in S\right\}\right). \\ K &:= N + \mathbb{R}_+\begin{pmatrix} 0_n \\ -1 \end{pmatrix}. \end{aligned}$$

Here N coincides with the second moment cone of the system θ_σ associated to σ, and K coincides with its characteristic cone. Indeed, when $\sigma \in \Omega_c$, $cl\,(K)$ is sometimes referred to as the *consequent relations cone* of σ (or θ_σ). Specifically, the so-called (non-homogeneous) *Farkas Lemma* ([27]) characterizes the linear inequalities $a'x \geq b$ which are consequences of the system $\sigma \in \Omega_c$ (i.e., inequalities which are satisfied at every feasible point of σ) as those satisfying $\binom{a}{b} \in cl\,(K)$.

If we introduce the cone $\mathbb{R}_+^{(T)}$ of all the functions $\lambda : T \to \mathbb{R}_+$ taking positive values only at finitely many points of T, $\binom{a}{b} \in cl\,(K)$ is equivalent to the existence of sequences $\{\lambda^r\} \subset \mathbb{R}_+^{(T)}$, $\{\mu^r\} \subset \mathbb{R}^S$ and $\{\gamma_r\} \subset \mathbb{R}_+$, such that

$$\begin{pmatrix} a \\ b \end{pmatrix} = \lim_r \left\{\sum_{t\in T}\lambda_t^r\begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s\in S}\mu_s^r\begin{pmatrix} a_s \\ b_s \end{pmatrix} + \gamma_r\begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}.$$

Theorem 1 *Given $\sigma \in \Omega$, and its associated $\theta_\sigma \in \Theta$, the following propositions hold*:

(i) $\sigma \in \Omega_c$ *if and only if* $\theta_\sigma \in \Theta_c$, *or equivalently* $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \notin cl\,(N)$;

(ii) $\sigma \in \Omega_w$ *if and only if* $\theta_\sigma \in \Theta_w$, *or equivalently* $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in cl\,(N)\setminus N$;

(iii) $\sigma \in \Omega_s$ *if and only if* $\theta_\sigma \in \Theta_s$, *or equivalently* $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in N$.

Furthermore, N can be replaced by K in the statements above.

Proof. The classification of θ_σ in terms of N (or K) can be traced out, for instance, from [12, Theorem 4.4] (see also [27]). The rest of the statement of the theorem is immediate. Just observe that if $\{a_t'x \geq b_t,\ t \in \widehat{T};\ a_s'x \geq b_s,\ s \in \widehat{S};\ -a_s'x \geq -b_s,\ s \in \widetilde{S}\}$ is an inconsistent finite subsystem of θ_σ, then $\{a_t'x \geq b_t, t \in \widehat{T};\ a_s'x \geq b_s,\ -a_s'x \geq -b_s,\ s \in \widehat{S} \cup \widetilde{S}\}$ is an inconsistent finite subsystem of σ. ■

In this paper we are interested in determining the distance to ill-posedness $d(\sigma, bd(\Omega_c))$ for $\sigma \in \Omega$, emphasizing the differences and similarities with $d(\theta_\sigma, bd(\Theta_c))$, and in general with $d(\theta, bd(\Theta_c))$ for $\theta \in \Theta$. The following remark points out that our parameter space Ω locally behaves as a normed space, despite the fact that some distances may be infinite when T is infinite.

Remark 1 Given $\varnothing \neq \widetilde{\Omega} \subsetneqq \Omega$ and $\sigma \notin \widetilde{\Omega}$, one has $d(\sigma, \widetilde{\Omega}) = d(\sigma, bd(\widetilde{\Omega}))$. In particular, if $\sigma \in \Omega_i$ then $d(\sigma, \Omega_c) = d(\sigma, bd(\Omega_c))$.

Note that this result is not true in general for metric spaces. Just consider $X := \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$ with the usual metric in $\mathbb{R}$, and $\widetilde{X} = \left\{\frac{1}{2n} \mid n \in \mathbb{N}\right\}$. Then $d(1, \widetilde{X}) = \frac{1}{2} < 1 = d(1, bd(\widetilde{X}))$, since $bd(\widetilde{X}) = \{0\}$.

Now we consider the following sets:

$$\begin{aligned} \Theta_\infty &:= \{\theta \in \Theta \mid d_\Theta(\theta, bd(\Theta_c)) = +\infty\}, \\ \Omega_\infty &:= \{\sigma \in \Omega \mid d_\Omega(\sigma, bd(\Omega_c)) = +\infty\}. \end{aligned}$$

Remark 2 Observe that Θ_∞ and Ω_∞ are formed by inconsistent systems, because any consistent system may be turned into an inconsistent one by means of a finite perturbation. Just replace an arbitrarily chosen inequality or equality by '$0_n'x \geq 1$' or '$0_n'x = 1$', respectively. Then the previous remark yields

$$\Theta_\infty \subset int(\Theta_i) \text{ and } \Omega_\infty \subset int(\Omega_i).$$

The set Θ_∞ may be characterized as follows (see [5, Proposition 1]), recalling the notation of (3) and (4).

Proposition 1 *A given system $\theta \in \Theta$ belongs to Θ_∞ if and only if there exists a sequence $\{\lambda^r\} \subset \mathbb{R}_+^{(\overline{T})}$ such that*

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \lim_r \sum_{t \in \overline{T}} \lambda_t^r \begin{pmatrix} a_t \\ b_t \end{pmatrix} \textit{ with } \lim_r \sum_{t \in \overline{T}} \lambda_t^r = 0.$$

The following result informs about the topological structure of $\Theta \backslash \Theta_\infty$.

Theorem 2 [5, Theorem 5] *Given $\theta \in \Theta \backslash \Theta_\infty$, one has*
(i) $\theta \in int(\Theta_i)$ *if and only if* $\theta \in int(\Theta_s)$;
(ii) $\theta \in int(\Theta_c)$ *if and only if* $\theta \in ext(\Theta_s)$;
(iii) $\theta \in bd(\Theta_c)$ *if and only if* $\theta \in bd(\Theta_s)$.

Condition (iii) in this theorem motivates the fact that, in [5], $bd(\Theta_s)$ is referred to as *generalized ill-posedness*, since it still constitutes a concept of ill-posedness in Θ_∞ : arbitrarily small perturbations of $\theta \in \Theta_\infty \cap bd(\Theta_s)$ may yield both, systems in Θ_s and systems in Θ_w.

The following theorem gathers some results traced out from [5] which describe the (generalized) well/ill-posedness of $\theta \in \Theta$, as well as the associated distance to ill-posedness in terms of the system's data. In this theorem we appeal to the so-called *hypographical set* associated to $\theta \in \Theta$, defined by

$$H(\theta) := C(\theta) + \mathbb{R}_+ \begin{pmatrix} 0_n \\ -1 \end{pmatrix}, \text{ where } C(\theta) := conv\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in \overline{T}\right\}\right), \qquad (5)$$

where we make use again of the notation (3) and (4).

Theorem 3 [5, Theorems 4 and 6] *Let $\theta \in \Theta$. Then, the following statements hold*:
(i) $\theta \in int(\Theta_s)$ *if and only if* $0_{n+1} \in int(H(\theta))$;
(ii) $\theta \in ext(\Theta_s)$ *if and only if* $0_{n+1} \in ext(H(\theta))$;
(iii) $\theta \in bd(\Theta_s)$ *if and only if* $0_{n+1} \in bd(H(\theta))$;
(iv) $d_\Theta(\theta, bd(\Theta_s)) = d(0_{n+1}, bd(H(\theta)))$.

Note that the left hand side in (iv) is a distance in the parameter space Θ (infinite-dimensional if T is infinite), whereas the right hand side is a distance in $\mathbb{R}^{n+1}$.

3 Stability of systems of linear equations and inequalities

In this section our aim is to analyze the stability of $\sigma \in \Omega$ in terms of its coefficient vectors. Specifically, characterizations for conditions '$\sigma \in int(\Omega_c)$', '$\sigma \in bd(\Omega_c)$', and consequently for '$\sigma \in int(\Omega_i)$', among others, are obtained, illustrating the structure of the parameter space Ω in relation to stability from the point of view of consistency/inconsistency. Some properties of $\sigma \in \Omega$ may be derived through the strategy of splitting each equation into two inequalities. In other words, there exists a direct relationship between some properties of σ and their counterparts for θ_σ, as we formalize in §3.1. Other properties will need *ad hoc* techniques, and they are gathered in §3.2.

Next, we synthesize the main results of the present section in the following theorem, which clarifies the topological structure of $\Omega \backslash \Omega_\infty$ in relation to consistency (recall that $\Omega_\infty \subset int(\Omega_i)$). Before that, we introduce the necessary ingredients. From now on, in order to simplify the notation, given $\sigma \in \Omega$ we shall denote

$$H := H(\theta_\sigma) \text{ and } C := C(\theta_\sigma). \qquad (6)$$

Note that, under our general assumption $S \neq \emptyset$, set C is given by

$$C = \left\{ \sum_{t \in T} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s \in S} \mu_s \begin{pmatrix} a_s \\ b_s \end{pmatrix} \,\middle|\, \lambda \in \mathbb{R}_+^{(T)}, \ \mu \in \mathbb{R}^S, \ \sum_{t \in T} \lambda_t + \sum_{s \in S} |\mu_s| \leq 1 \right\}. \qquad (7)$$

This expression can easily be derived from the fact that $0_{n+1} \in C$ (due to the presence of equations) and the convexity of C. Analyzing the stability of $\sigma \in \Omega$ will require as additional ingredients the following sets:

$$E := \left\{ \sum_{t \in T} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s \in S} \mu_s \begin{pmatrix} a_s \\ b_s \end{pmatrix} \middle| \; \lambda \in \mathbb{R}_+^{(T)}, \; \mu \in \mathbb{R}^S, \; \sum_{t \in T} \lambda_t + \sum_{s \in S} |\mu_s| = 1 \right\} \tag{8}$$

and

$$G := E + \mathbb{R}_+ \begin{pmatrix} 0_n \\ -1 \end{pmatrix}. \tag{9}$$

Theorem 4 *Let $\sigma \in \Omega \backslash \Omega_\infty$. Then the following statements hold*:
(i) $\sigma \in int\,(\Omega_i)$ *if and only if* $\sigma \in int\,(\Omega_s)$, *or equivalently*

$$0_{n+1} \in int\,(H)\,;$$

(ii) $\sigma \in int\,(\Omega_c)$ *if and only if* $\sigma \in ext\,(\Omega_s)$, *or equivalently*

$$0_{n+1} \in bd\,(H) \backslash bd\,(G)\,;$$

(iii) $\sigma \in bd\,(\Omega_c)$ *if and only if* $\sigma \in bd\,(\Omega_s)$, *or equivalently*

$$0_{n+1} \in bd\,(H) \cap bd\,(G)\,.$$

Proof. (i) comes from Theorem 3 and Corollary 2, (iii) gathers Theorems 6 and 8, and (ii) is a straightforward consequence of (i) and (iii). ■

The previous theorem provides the counterpart of Theorems 2 and 3 for linear systems in presence of equations. Observe that the situation concerning Theorem 3 does not exhibit such an exact analogy as with Theorem 2. To start with, note that the presence of equations entails always $0_{n+1} \in H$.

Next we provide two examples in order to illustrate these discrepancies. Example 1 shows that condition (i) in Theorem 3, which admits a direct translation in $\Omega \backslash \Omega_\infty$ (Theorem 4(i)), cannot be extended to Ω_∞. Example 2 provides a system in $int\,(\Omega_c)$, and then with $0_{n+1} \in bd\,(H) \backslash bd\,(G)$, which will be used later on to illustrate some of the main results of the paper.

Example 1 Consider the system, in $\mathbb{R}$, $\sigma = \{x \geq t, \; t \in \mathbb{N}; \; 0x = 1\} \in \Omega_\infty$, which obviously satisfies $0_{n+1} \in bd\,(H)$. The reader can check that $\sigma \in int\,(\Omega_s)$, since $\binom{0}{1} \in N_1$ (see Theorem 1) for any $\sigma_1 \in \Omega$ such that $d\,(\sigma_1, \sigma) < 1$ (with respect to the supremum norm in $\mathbb{R}^2$). Specifically, two cases arise: if the coefficient of x in the perturbed equation is non-zero, then $\binom{0}{1} \in int\,(N_1)$; and if this coefficient is zero, then the perturbed equation is $0x = \rho$ with $\rho > 0$. On the other hand, $\theta_\sigma \notin int\,(\Theta_s)$, since the perturbed systems $\theta_\varepsilon = \{x \geq t, \; t \in \mathbb{N}; \; \varepsilon x \geq 1, \; \varepsilon x \geq -1\}$, with $\varepsilon > 0$, belong to Θ_w.

Example 2 Consider the system, in $\mathbb{R}^2$, $\sigma = \{x_1 + x_2 \geq 0, \; x_1 - x_2 \geq 0; \; x_2 = 0\}$. It is evident that $0_{n+1} \in bd\,(H)$ (yielding $\theta_\sigma \in bd\,(\Theta_c)$ according to Theorems 2 and 3) and the reader can easily check, by means of direct arguments, that $\sigma \in int\,(\Omega_c)$. Figure 1 below illustrates sets C, E, H and G in this example and shows that, in general, $bd\,(G) \not\subset bd\,(H)$.

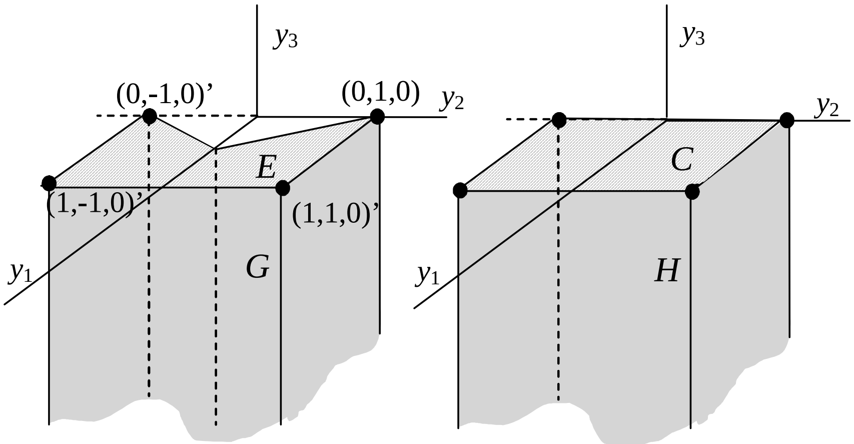

Figure 1: Illustration of Example 2

We will return later to this example in order to calculate $d(\sigma, bd(\Omega_c))$. At this moment we emphasize the fact that, under the assumption $S \neq \emptyset$, sets E and G are, in general, nonconvex.

3.1 Stability via splitting equations into inequalities

The following result yields in particular the equivalence between '$\sigma \in int(\Omega_i)$' and '$\theta_\sigma \in int(\Theta_i)$'. As Example 2 shows, the counterpart for consistent systems does not hold.

Theorem 5 *Let* $\sigma = \{a_t' x \geq b_t,\ t \in T;\ a_s' x = b_s,\ s \in S\} \in \Omega$, *and consider the associated* $\theta_\sigma \in \Theta$. *Then we have*

$$d_\Omega(\sigma, \Omega_c) = d_\Theta(\theta_\sigma, \Theta_c).$$

Proof. Let us start with the inequality $d(\sigma, \Omega_c) \geq d(\theta_\sigma, \Theta_c)$, assuming the non-trivial case $d(\sigma, \Omega_c) < +\infty$. Take a sequence $\{\sigma_r\} \subset \Omega_c$ such that $d(\sigma, \Omega_c) = \lim_r d(\sigma, \sigma_r)$. Since, for all $r \in \mathbb{N}$, we have $\theta_{\sigma_r} \in \Theta_c$ and $d(\theta_\sigma, \theta_{\sigma_r}) = d(\sigma, \sigma_r)$, the aimed inequality holds.

Now let us see that $d(\sigma, \Omega_c) \leq d(\theta_\sigma, \Theta_c)$, now under the non-trivial case $d(\theta_\sigma, \Theta_c) < +\infty$. Consider a sequence $\{\theta_r\} \subset \Theta_c$ such that $d(\theta_\sigma, \Theta_c) = \lim_r d(\theta_\sigma, \theta_r)$. We will construct an appropriate $\{\sigma_r\} \subset \Omega_c$. Write, for each $r \in \mathbb{N}$,

$$\theta_r := \left\{ (a_t^r)' x \geq b_t^r,\ t \in T;\ (a_{s1}^r)' x \geq b_{s1}^r,\ (a_{s2}^r)' x \geq b_{s2}^r,\ s \in S \right\},$$

and take feasible points x^r for every θ_r. Then we have, for each $s \in S$,

$$\begin{array}{rcl} a_s' x^r + (a_{s1}^r - a_s)' x^r - b_s - (b_{s1}^r - b_s) & \geq & 0, \\ a_s' x^r - (a_{s2}^r + a_s)' x^r - b_s + (b_{s2}^r + b_s) & \leq & 0. \end{array}$$

Therefore, for each r and each s, there exists $\alpha_s^r \in [0,1]$ such that

$$\begin{array}{l}\alpha_s^r \left[a_s' x^r + (a_{s1}^r - a_s)' x^r - b_s - (b_{s1}^r - b_s)\right] + \\ + (1-\alpha_s^r)\left[a_s' x^r - (a_{s2}^r + a_s)' x^r - b_s + (b_{s2}^r + b_s)\right] = 0.\end{array}$$

In other words, for all r we conclude that x^r is also a feasible point of the system

$$\sigma_r := \left\{(a_t^r)' x \geq b_t^r,\ t \in T;\ \ (a_s^r)' x = b_s^r,\ s \in S\right\}$$

where

$$\begin{array}{l}a_s^r := a_s + \alpha_s^r (a_{s1}^r - a_s) + (1-\alpha_s^r)(-a_{s2}^r - a_s), \\ b_s^r := b_s + \alpha_s^r (b_{s1}^r - b_s) + (1-\alpha_s^r)(-b_{s2}^r - b_s).\end{array}$$

Moreover, for all r we have

$$d(\sigma, \sigma_r) \leq d(\theta_\sigma, \theta_r)$$

due to the facts

$$\sup_{t\in T} \left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} - \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} \right\| \leq d(\theta_\sigma, \theta_r),$$

and

$$\begin{array}{rcl}\max\limits_{s\in S} \left\| \begin{pmatrix} a_s \\ b_s \end{pmatrix} - \begin{pmatrix} a_s^r \\ b_s^r \end{pmatrix} \right\| & \leq & \max\limits_{s\in S} \left\{ \alpha_s^r \left\| \begin{pmatrix} a_{s1}^r - a_s \\ b_{s1}^r - b_s \end{pmatrix} \right\| + (1-\alpha_s^r) \left\| \begin{pmatrix} a_{s2}^r + a_s \\ b_{s2}^r + b_s \end{pmatrix} \right\| \right\} \\ & \leq & \alpha_s^r d(\theta_\sigma, \theta_r) + (1-\alpha_s^r) d(\theta_\sigma, \theta_r) = d(\theta_\sigma, \theta_r).\end{array}$$

In this way, $d(\sigma, \Omega_c) \leq \liminf_r d(\sigma, \sigma_r) \leq \lim_r d(\theta_\sigma, \theta_r) = d(\theta_\sigma, \Theta_c)$. ■

Corollary 1 *Let $\sigma \in \Omega$ and consider the associated $\theta_\sigma \in \Theta$. Then the following statements hold*:
(i) $\sigma \in cl(\Omega_c)$ *if and only if* $\theta_\sigma \in cl(\Theta_c)$;
(ii) $\sigma \in \Omega_\infty$ *if and only if* $\theta_\sigma \in \Theta_\infty$.

Proof. (i) come straightforwardly from Theorem 5. For (ii) take also Remark 1 into account. ■

Corollary 2 *Let $\sigma \in \Omega \backslash \Omega_\infty$. The following conditions are equivalent*:
(i) $\sigma \in int(\Omega_i)$;
(ii) $\theta_\sigma \in int(\Theta_i)$;
(iii) $\theta_\sigma \in int(\Theta_s)$;
(iv) $\sigma \in int(\Omega_s)$.

Proof. (i)⇔(ii) comes straightforwardly from statement (i) in the previous corollary. Theorem 2 yields (ii)⇔(iii). (iii)⇒(iv) is immediate as far as small perturbations of σ in Ω can be viewed as particular small perturbations of θ_σ in Θ. Finally, (iv)⇒(i) is trivial. ■

Corollary 3 *The following conditions hold*:

(i) $int(\Omega_i) = \Omega_\infty \cup int(\Omega_s)$;

(ii) *Given* $\sigma \in \Omega$, *one has* $\sigma \in \Omega_\infty$ *if and only if there exists* $\{\lambda^r\} \subset \mathbb{R}_+^{(T)}$ *such that*

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \lim_r \sum_{t\in T} \lambda_t^r \begin{pmatrix} a_t \\ b_t \end{pmatrix} \text{ with } \lim_r \sum_{t\in T} \lambda_t^r = 0. \tag{10}$$

Hence, $\sigma \in int(\Omega_i)$ *if and only if either* $0_{n+1} \in int(H)$ *or* (10) *holds for some* $\{\lambda^r\} \subset \mathbb{R}_+^{(T)}$.

Proof. (i) is a straightforward consequence of the previous corollary together with Remark 2. (ii) comes from Corollary 1 and Proposition 1, taking into account that we can remove the finite (and hence bounded) set $\{\binom{a_s}{b_s},\ s \in S\}$ from the expression of $\binom{0_n}{1}$ in that proposition because there we have $\sum_{s\in S} \lambda_s^r \to 0$ as $r \to \infty$. ■

3.2 Specifics in presence of equations

The strategy of splitting equations into inequalities has allowed us to characterize '$\sigma \in cl(\Omega_c)$' in terms of the system's data, but this strategy gives no information about whether '$\sigma \in int(\Omega_c)$' or '$\sigma \in bd(\Omega_c)$'. This is the main concern of the present subsection.

The characterization of condition '$\sigma \in bd(\Omega_c)$' provided here makes use of sets H, E and G, associated to $\sigma \in \Omega$, defined in (6), (8) and (9). Set E was already introduced in [2], where it is a key ingredient in the expressions providing the modulus and the radius of metric regularity of a set-valued mapping associated to systems in the form (1). See Section 5 for details.

Proposition 2 *Let* $\sigma \in \Omega$. *Then* $G = H$ *if and only if* $0_{n+1} \in G$.

Proof. The 'only if' part is a straightforward consequence of the fact that $0_{n+1} \in H$ in presence of equations. Suppose now $0_{n+1} \in G$ and write

$$0_{n+1} = \sum_{t\in T} \eta_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s\in S} \nu_s \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \gamma \begin{pmatrix} 0_n \\ -1 \end{pmatrix}$$

with $\eta \in \mathbb{R}_+^{(T)}$, $\nu \in \mathbb{R}^S$, $\sum_{t\in T} \eta_t + \sum_{s\in S} |\nu_s| = 1$, and $\gamma \geq 0$. Take any

$$\begin{pmatrix} a \\ b \end{pmatrix} := \sum_{t\in T} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s\in S} \mu_s \begin{pmatrix} a_s \\ b_s \end{pmatrix} \in C,$$

where $\lambda \in \mathbb{R}_+^{(T)}$, $\mu \in \mathbb{R}^S$ and $\sum_{t\in T} \lambda_t + \sum_{s\in S} |\mu_s| \leq 1$ (see 7); and let us see that $\begin{pmatrix} a \\ b \end{pmatrix} \in G$, which will finish the proof. To do this observe that there exists $\alpha \geq 0$ such that

$$\phi(\alpha) := \sum_{t\in T} (\lambda_t + \alpha\eta_t) + \sum_{s\in S} |\mu_s + \alpha\nu_s| = 1, \tag{11}$$

which is a consequence of the continuity of ϕ on $\mathbb{R}_+$ together with $\phi(0) \leq 1$ and $lim_{\alpha \to +\infty} \phi(\alpha) = +\infty$.

Then the expression

$$\begin{pmatrix} a \\ b \end{pmatrix} + \alpha 0_{n+1} = \sum_{t \in T} (\lambda_t + \alpha \eta_t) \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s \in S} (\mu_s + \alpha \nu_s) \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \alpha \gamma \begin{pmatrix} 0_n \\ -1 \end{pmatrix}$$

ensures, appealing to (11), that $\begin{pmatrix} a \\ b \end{pmatrix} \in G$. ■

The following theorem characterizes condition '$\sigma \in bd(\Omega_s)$' provided that $\sigma \in \Omega \backslash \Omega_\infty$. This theorem, together with Theorem 8, yields the aimed characterization of $bd(\Omega_c)$.

Theorem 6 *Let $\sigma \in \Omega \backslash \Omega_\infty$. Then, $\sigma \in bd(\Omega_s)$ if and only if $0_{n+1} \in bd(H) \cap bd(G)$.*

Proof. Let $\sigma \in bd(\Omega_s)$. In particular, there exists a sequence $\{\sigma_r\} \subset \Omega_s$ converging to σ. Then, from Theorem 1, for each $r \in \mathbb{N}$ there must exist $\lambda^r \in \mathbb{R}_+^{(T)}$ and $\mu^r \in \mathbb{R}^S$ such that

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \sum_{t \in T} \lambda_t^r \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} + \sum_{s \in S} \mu_s^r \begin{pmatrix} a_s^r \\ b_s^r \end{pmatrix}.$$

Taking $\gamma_r := \sum_{t \in T} \lambda_t^r + \sum_{s \in S} |\mu_s^r| > 0$, we have

$$0_{n+1} = \sum_{t \in T} \frac{\lambda_t^r}{\gamma_r} \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} + \sum_{s \in S} \frac{\mu_s^r}{\gamma_r} \begin{pmatrix} a_s^r \\ b_s^r \end{pmatrix} + \frac{1}{\gamma_r} \begin{pmatrix} 0_n \\ -1 \end{pmatrix}, \tag{12}$$

with

$$\sum_{t \in T} \frac{\lambda_t^r}{\gamma_r} + \sum_{s \in S} \left| \frac{\mu_s^r}{\gamma_r} \right| = 1. \tag{13}$$

From (12) we obtain

$$\begin{aligned} \left\| \sum_{t \in T} \frac{\lambda_t^r}{\gamma_r} \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s \in S} \frac{\mu_s^r}{\gamma_r} \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \frac{1}{\gamma_r} \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\| &= \\ \left\| \sum_{t \in T} \frac{\lambda_t^r}{\gamma_r} \left(\begin{pmatrix} a_t \\ b_t \end{pmatrix} - \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} \right) + \sum_{s \in S} \frac{\mu_s^r}{\gamma_r} \left(\begin{pmatrix} a_s \\ b_s \end{pmatrix} - \begin{pmatrix} a_s^r \\ b_s^r \end{pmatrix} \right) \right\| &\leq \\ \sum_{t \in T} \frac{\lambda_t^r}{\gamma_r} d(\sigma, \sigma_r) + \sum_{s \in S} \left| \frac{\mu_s^r}{\gamma_r} \right| d(\sigma, \sigma_r) &= d(\sigma, \sigma_r). \end{aligned}$$

Appealing to (13), we observe that $\sum_{t \in T} \frac{\lambda_t^r}{\gamma_r} \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s \in S} \frac{\mu_s^r}{\gamma_r} \begin{pmatrix} a_s \\ b_s \end{pmatrix} \in E$ and then

$$0_{n+1} = \lim_r \left\{ \sum_{t \in T} \frac{\lambda_t^r}{\gamma_r} \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s \in S} \frac{\mu_s^r}{\gamma_r} \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \frac{1}{\gamma_r} \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \in cl(G).$$

From Theorem 3 and Corollary 2, we have $0_{n+1} \notin int\,(H)$ and so $0_{n+1} \notin int\,(G)$ because $G \subset H$. Then, $0_{n+1} \in bd\,(H) \cap bd\,(G)$ (recalling again that 0_{n+1} always belongs to H when $S \neq \emptyset$).

Conversely, assume that $0_{n+1} \in bd\,(H) \cap bd\,(G)$, and take any sequence $\left\{\begin{pmatrix} a^r \\ b^r \end{pmatrix}\right\} \subset G$ converging to 0_{n+1}. For each $r \in \mathbb{N}$ there must exist $\lambda^r \in \mathbb{R}_+^{(T)}$ and $\mu^r \in \mathbb{R}^S$, with $\sum_{t\in T} \lambda_t^r + \sum_{s\in S} |\mu_s^r| = 1$, and $\gamma_r \geq 0$ such that

$$\begin{pmatrix} a^r \\ b^r \end{pmatrix} = \sum_{t\in T} \lambda_t^r \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s\in S} \mu_s^r \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \gamma_r \begin{pmatrix} 0_n \\ -1 \end{pmatrix}.$$

Hence we obtain

$$\begin{aligned} \left(\gamma_r + \frac{1}{r}\right) \begin{pmatrix} 0_n \\ 1 \end{pmatrix} &= \sum_{t\in T} \lambda_t^r \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s\in S} \mu_s^r \begin{pmatrix} a_s \\ b_s \end{pmatrix} \\ &\quad - \left(\sum_{t\in T} \lambda_t^r + \sum_{s\in S} |\mu_s^r|\right) \left[\begin{pmatrix} a^r \\ b^r \end{pmatrix} - \frac{1}{r} \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right] \end{aligned}$$

and, so,

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \sum_{t\in T} \frac{\lambda_t^r}{\gamma_r + \frac{1}{r}} \begin{pmatrix} a_t - a^r \\ b_t - b^r + \frac{1}{r} \end{pmatrix} + \sum_{s\in S} \frac{\mu_s^r}{\gamma_r + \frac{1}{r}} \begin{pmatrix} a_s - sign\,(\mu_s^r)\, a^r \\ b_s - sign\,(\mu_s^r)\, b^r + sign\,(\mu_s^r)\, \frac{1}{r} \end{pmatrix}.$$

Now we consider, for each $r \in \mathbb{N}$, the system

$$\sigma_r := \left\{ \begin{array}{rcl} (a_t - a^r)' x & \geq & b_t - b^r + \frac{1}{r}\,,\ t \in T; \\ (a_s - sign\,(\mu_s^r)\, a^r)' x & = & b_s - sign\,(\mu_s^r)\, b^r + sign\,(\mu_s^r)\, \frac{1}{r},\ s \in S \end{array} \right\}.$$

So we have $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in N_r$, where N_r is the second moment cone of σ_r. Hence, Theorem 1 states $\sigma_r \in \Omega_s$ for all r and, since $\{\sigma_r\}$ clearly converges to σ, because $\lim_r \begin{pmatrix} a^r \\ b^r \end{pmatrix} = 0_{n+1}$, we obtain $\sigma \in cl\,(\Omega_s)$. Since $0_{n+1} \notin int\,(H)$, it must be $\sigma \in bd\,(\Omega_s)$. ■

The following theorem shows one of the essential differences between the well-posedness with respect to consistency in Ω and in Θ. Theorem 6.1 in [12] shows that condition $\theta \in int\,(\Theta_c)$, provided that $\theta \in \Theta_c$, turns out to be equivalent to different stability criteria spread out in the literature. One of these conditions is expressed by $0_{n+1} \notin cl\,(C)$. Note that set E plays the role of C when we deal with Ω.

Theorem 7 *Let $\sigma \in \Omega_c$. Then $\sigma \in int\,(\Omega_c)$ if and only if $0_{n+1} \notin cl\,(E)$.*

Proof. We begin by supposing $0_{n+1} \notin cl\,(E)$. Write $\sigma := \{a_t' x \geq b_t,\ t \in T;\ a_s' x = b_s,\ s \in S\} \in \Omega_c$. Denote by $\widetilde{\Theta}$ the parameter space of all inequality systems indexed by $T \cup S$. Given a 'choice of signs' $\gamma \in \{-1,1\}^S$, we consider the system

$$\theta_\gamma := \{a_t' x \geq b_t,\ t \in T;\ \gamma(s)\, a_s' x \geq \gamma(s)\, b_s,\ s \in S\} \in \widetilde{\Theta}_c. \tag{14}$$

For any γ we have

$$C_\gamma := conv\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T;\ \begin{pmatrix} \gamma(s)\, a_s \\ \gamma(s)\, b_s \end{pmatrix},\ s \in S \right\} \subset E.$$

In fact,

$$E = \bigcup_{\gamma \in \{-1,1\}^S} C_\gamma.$$

Under the current hypothesis, for every γ one has $0_{n+1} \notin cl\,(C_\gamma)$, which together with the consistency of θ_γ entails, applying [13, Theorem 3.1], $\theta_\gamma \in int(\widetilde{\Theta}_c)$.

Assume, reasoning by contradiction, that $\sigma \in cl\,(\Omega_i)$ and write $\sigma = \lim_r \sigma_r$, with $\sigma_r := \left\{ (a_t^r)' x \geq b_t^r,\ t \in T;\ (a_s^r)' x = b_s^r,\ s \in S \right\} \in \Omega_i$ for all r. We shall find $\theta_{\gamma_0} \in cl(\widetilde{\Theta}_i)$ for a suitable $\gamma_0 \in \{-1,1\}^S$, attaining a contradiction. According to Theorem 1, write, for each r,

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \lim_k \left\{ \sum_{t \in T} \lambda_t^{r,k} \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} + \sum_{s \in S} \mu_s^{r,k} \begin{pmatrix} a_s^r \\ b_s^r \end{pmatrix} \right\}, \tag{15}$$

where $\left\{\lambda^{r,k}\right\}_{k \in \mathbb{N}} \subset \mathbb{R}_+^{(T)}$ and $\left\{\mu^{r,k}\right\}_{k \in \mathbb{N}} \subset \mathbb{R}^S$. Because of the finiteness of S, and considering suitable subsequences if necessary, we may assume that S is partitioned as $S_+ \cup S_-$ with

$$\mu_s^{r,k} \geq 0 \text{ if } s \in S_+ \text{ and } \mu_s^{r,k} < 0 \text{ if } s \in S_-, \text{ for all } r \text{ and all } k.$$

In other words, the sets $S_+ := \{s \in S \mid \mu_s^{r,k} \geq 0\}$ and $S_- := \{s \in S \mid \mu_s^{r,k} < 0\}$ do not depend on either r or k. Define $\gamma_0 \in \{-1,1\}^S$ by $\gamma_0(s) = 1$ if $s \in S_+$ and $\gamma_0(s) = -1$ if $s \in S_-$. Then (15) yields, for each r,

$$\theta_{\gamma_0}^r := \left\{ (a_t^r)' x \geq b_t^r,\ t \in T;\ \gamma_0(s)\,(a_s^r)' x \geq \gamma_0(s)\, b_s^r,\ s \in S \right\} \in \widetilde{\Theta}_i,$$

appealing to [12, Theorem 4.4] (the version of Theorem 1 for inequality systems). Thus $\theta_{\gamma_0} = \lim_r \theta_{\gamma_0}^r \in cl(\widetilde{\Theta}_i)$.

Conversely, we assume now that $\sigma \in int\,(\Omega_c)$, and so, by Theorem 3 and Corollary 2, one has $0_{n+1} \notin int\,(H)$ and then $0_{n+1} \notin int\,(G)$ (recall that $G \subset H$). Since $0_{n+1} \in bd\,(H)$, Theorem 6 ensures $0_{n+1} \notin bd\,(G)$. So, $0_{n+1} \notin cl\,(G)$ and in particular $0_{n+1} \notin cl\,(E)$. ■

In order to go deeply in the topology relative to $\Omega \backslash \Omega_\infty$, we need the following technical result.

Lemma 1 *Let $\sigma \in \Omega_i \backslash \Omega_\infty$. Then, there exists $\rho \geq 0$ such that* $\begin{pmatrix} 0_n \\ \rho \end{pmatrix} \in cl\,(E)$.

Proof. Since $\binom{0_n}{1} \in cl\,(N)$ (Theorem 1), there exist some sequences $\{\lambda^r\} \subset \mathbb{R}_+^{(T)}$ and $\{\mu^r\} \subset \mathbb{R}^S$ such that $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \lim_r \left\{ \sum_{t\in T} \lambda_t^r \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s\in S} \mu_s^r \begin{pmatrix} a_s \\ b_s \end{pmatrix} \right\}$. Then, the sequence $\{\eta_r\}$, where $\eta_r := \sum_{t\in T} \lambda_t^r + \sum_{s\in S} |\mu_s^r|$ for all r, does not converge to zero, because $\sigma \notin \Omega_\infty$ (appealing to Corollary 1(ii) and Proposition 1). Therefore, $\{\eta_r\}$ has a subsequence, denoted in the same way for the sake of brevity, such that $\eta_r > 0$ for all r and $\{1/\eta_r\}$ converges to certain $\rho \geq 0$. Then $\begin{pmatrix} 0_n \\ \rho \end{pmatrix} = \lim_r \left\{ \sum_{t\in T} \frac{\lambda_t^r}{\eta_r} \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s\in S} \frac{\mu_s^r}{\eta_r} \begin{pmatrix} a_s \\ b_s \end{pmatrix} \right\} \in cl\,(E)$. ■

Theorem 8 *Let $\sigma \in \Omega\backslash\Omega_\infty$. Then, $\sigma \in bd\,(\Omega_c)$ if and only if $\sigma \in bd\,(\Omega_s)$.*

Proof. First we prove the 'if part'. If $\sigma \in bd\,(\Omega_s)$, obviously $\sigma \notin int\,(\Omega_c)$. Moreover, Corollary 2 ensures $\sigma \notin int\,(\Omega_i)$. Thus, $\sigma \in bd\,(\Omega_c)$. In order to prove the 'only if part', assume that $\sigma \in bd\,(\Omega_c)$. If $\sigma \in \Omega_c$, the previous theorem yields $0_{n+1} \in cl(E) \subset cl\,(G)$. Now Corollary 2 and Theorem 3 entail $0_{n+1} \notin int\,(G)$ because $0_{n+1} \notin int\,(H)$. In such a way, $0_{n+1} \in bd\,(H) \cap bd\,(G)$ and Theorem 6 guarantees $\sigma \in bd\,(\Omega_s)$. In other case, if $\sigma \notin \Omega_c$, the previous Lemma yields the existence of $\rho \geq 0$ such that $\begin{pmatrix} 0_n \\ \rho \end{pmatrix} \in cl\,(E)$. Then $0_{n+1} \in cl(E) + \rho \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \subset cl\,(G)$ and, as in the previous case, $\sigma \in bd\,(\Omega_s)$. ■

Corollary 4 *The following statements hold:*
(i) $int\,(\Omega_w) \subset \Omega_\infty$
(ii) $(\Omega_w\backslash\Omega_\infty) \subset bd\,(\Omega_c)$

Proof. (i) We have $int\,(\Omega_w) \cap (\Omega\backslash\Omega_\infty) \subset [int\,(\Omega_i) \backslash int\,(\Omega_s)] \cap (\Omega\backslash\Omega_\infty) = \varnothing$, according to Corollary 2.

(ii) Let $\sigma \in (\Omega_w\backslash\Omega_\infty)$. Appealing to Theorem 1(i) and Corollary 1(ii) we obtain $\theta_\sigma \in (\Theta_w\backslash\Theta_\infty)$, and then $\theta_\sigma \notin int\,(\Theta_i)$, according to Theorem 2. So, from Theorem 5, $d\,(\sigma, \Omega_c) = d\,(\theta_\sigma, \Theta_c) = 0$. Therefore, $\sigma \in bd\,(\Omega_c)$. ■

4 Distance to ill-posedness

In this section we extend Theorem 3(iv), established for linear inequality systems, to systems of linear equations and inequalities. First, we need the following lemma.

Lemma 2 *Let $\sigma, \tilde{\sigma} \in \Omega$ such that $d\,(\sigma, \tilde{\sigma}) \leq \varepsilon$ for some $\varepsilon > 0$, and assume that there exists $\rho \geq \varepsilon$ verifying $\rho cl\,(B) \cap cl\,(G) = \varnothing$. Then we have $(\rho - \varepsilon)\, cl\,(B) \cap cl\left(\tilde{G}\right) = \varnothing$. Consequently, if $d\,(0_{n+1}, cl\,(G)) > \rho$ then we have $d\left(0_{n+1}, cl\left(\tilde{G}\right)\right) > \rho - \varepsilon$.*

Proof. Suppose, reasoning by contradiction, that there exists $y \in (\rho - \varepsilon)\, cl\,(B) \cap cl\left(\tilde{G}\right)$. Then, there exist some sequences $\{\lambda^r\} \subset \mathbb{R}_+^{(T)}, \{\mu^r\} \subset \mathbb{R}^S$ and $\{\gamma_r\} \subset \mathbb{R}_+$, with $\sum_{t\in T} \lambda_t^r + \sum_{s\in S} |\mu_s^r| = 1$ for all r, verifying

$$y = \lim_r y^r, \text{ where } y^r := \sum_{t\in T} \lambda_t^r \begin{pmatrix} \tilde{a}_t \\ \tilde{b}_t \end{pmatrix} + \sum_{s\in S} \mu_s^r \begin{pmatrix} \tilde{a}_s \\ \tilde{b}_s \end{pmatrix} + \gamma_r \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \tilde{G}.$$

Define, for each r,

$$x^r := \sum_{t\in T} \lambda_t^r \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s\in S} \mu_s^r \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \gamma_r \begin{pmatrix} 0_n \\ -1 \end{pmatrix}.$$

Then $\|x^r - y^r\| \leq \left(\sum_{t\in T} \lambda_t^r + \sum_{s\in S} |\mu_s^r|\right) d\,(\sigma, \tilde{\sigma}) = d\,(\sigma, \tilde{\sigma}) \leq \varepsilon$ and, so, $\|x^r\| \leq \|y^r\| + \varepsilon$. Assuming w.l.o.g. that $\{x^r\}$ converges to certain $x \in cl\,(G)$, we have $\|x\| \leq \|y\| + \varepsilon \leq \rho$, attaining a contradiction. ■

Theorem 9 *Let $\sigma \in \Omega \backslash \Omega_\infty$. Then*

$$d\,(\sigma, bd\,(\Omega_c)) = d\,(\sigma, bd\,(\Omega_s)) = d\,(0_{n+1}, bd\,(G)).$$

Proof. The equality $d\,(\sigma, bd\,(\Omega_c)) = d\,(\sigma, bd\,(\Omega_s))$ comes from Theorem 4(iii), taking also into account that any system whose distance to σ is finite remains in $\Omega \backslash \Omega_\infty$. In order to prove $d\,(\sigma, bd\,(\Omega_s)) = d\,(0_{n+1}, bd\,(G))$ we shall distinguish three cases.

Case 1: $\sigma \in int\,(\Omega_s)$. In this case Theorem 4(i) guarantees $0_{n+1} \in int\,(H)$. So, for $\varepsilon > 0$ small enough we can write

$$\begin{pmatrix} 0_n \\ \varepsilon \end{pmatrix} = \sum_{t\in T} \eta_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s\in S} \nu_s \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \alpha \begin{pmatrix} 0_n \\ -1 \end{pmatrix}, \tag{16}$$

for certain $\eta \in \mathbb{R}_+^{(T)}$, $\nu \in \mathbb{R}^S$ and $\alpha \geq 0$, with $\beta := \sum_{t\in T} \eta_t + \sum_{s\in S} |\nu_s| \leq 1$ (see (7)). Note that $\varepsilon > 0$ implies $\beta > 0$, and (16) may be rewritten as

$$0_{n+1} = \sum_{t\in T} \frac{\eta_t}{\beta} \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s\in S} \frac{\nu_s}{\beta} \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \frac{\alpha + \varepsilon}{\beta} \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in G.$$

Now Proposition 2 entails $G = H$. Moreover, appealing to Theorem 5 and Theorem 3 (see also Remark 1 and Theorem 4), we have

$$d\,(\sigma, bd\,(\Omega_s)) = d\,(\sigma, \Omega_c) = d\,(\theta_\sigma, \Theta_c) = d\,(0_{n+1}, bd\,(H)) = d\,(0_{n+1}, bd\,(G)).$$

Case 2: $\sigma \in int\,(\Omega_c)$. Observe that, in this case, $G \neq \mathbb{R}^{n+1}$, because otherwise $0_{n+1} \in int\,(H)$. So, $bd\,(G)$ is a nonempty closed set and

$$d\,(0_{n+1}, bd\,(G)) = \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| \tag{17}$$

for certain $\begin{pmatrix} a \\ b \end{pmatrix} \in bd\,(G)$.

Next we prove that $d\,(\sigma, bd\,(\Omega_s)) \geq \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|$.

Reasoning by contradiction, we suppose that there exist some $\tilde{\sigma} \in bd\,(\Omega_s)$, with $d\,(\sigma, \tilde{\sigma}) \leq \rho < \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|$ for certain $\rho > 0$. Since $\sigma \in int\,(\Omega_c)$ then, from Theorem 4(ii), $0_{n+1} \in bd\,(H) \backslash bd\,(G)$, and so $0_{n+1} \notin cl\,(G)$. Therefore, using (17), we obtain $d\,(0_{n+1}, cl\,(G)) = \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|$. So, $\rho cl\,(B) \cap cl\,(G) = \varnothing$ and applying the previous lemma (with $\varepsilon = \rho$) we conclude $\{0_{n+1}\} \cap cl\left(\tilde{G}\right) = \varnothing$.

Hence $0_{n+1} \in ext\left(\tilde{G}\right)$, in contradiction with $0_{n+1} \in bd\left(\tilde{H}\right) \cap bd\left(\tilde{G}\right)$, which comes from the assumption $\tilde{\sigma} \in bd\,(\Omega_s) \cap (\Omega \backslash \Omega_\infty)$ and Theorem 4(iii).

In order to prove that $d\,(\sigma, bd\,(\Omega_s)) \leq \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|$ we will construct a system $\tilde{\sigma} \in cl\,(\Omega_s)$ verifying $d\,(\sigma, \tilde{\sigma}) \leq \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|$.

Since $\begin{pmatrix} a \\ b \end{pmatrix} \in bd\,(G)$, there exist some sequences $\{\lambda^r\} \subset \mathbb{R}_+^{(T)}, \{\mu^r\} \subset \mathbb{R}^S$, $\{\gamma_r\} \subset \mathbb{R}_+$ with $\sum_{t \in T} \lambda_t^r + \sum_{s \in S} |\mu_s^r| = 1$ for all r, verifying

$$\begin{pmatrix} a \\ b \end{pmatrix} = \lim_r \left\{ \sum_{t \in T} \lambda_t^r \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{s \in S} \mu_s^r \begin{pmatrix} a_s \\ b_s \end{pmatrix} + \gamma_r \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}. \tag{18}$$

From this expression, we can define S_+ and S_- in a similar way as in the proof of Theorem 7. That is, $S = S_+ \cup S_-$ (with $S_+ \cap S_- = \varnothing$) and w.l.o.g. $\{\mu_s^r\}_r \subset \mathbb{R}_+$ if $s \in S_+$ and $\{\mu_s^r\}_r \subset \mathbb{R} \setminus \mathbb{R}_+$ if $s \in S_-$.

Now we consider the system:

$$\tilde{\sigma} := \left\{ \begin{array}{c} (a_t - a)' x \geq b_t - b, \ t \in T \\ (a_s - a)' x = b_s - b, \ s \in S_+; \ \ (b_s + a)' x = b_s + b, \ s \in S_- \end{array} \right\}. \tag{19}$$

Recall that $d\,(\sigma, \tilde{\sigma}) = \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| < +\infty$ and thus $\tilde{\sigma} \notin \Omega_\infty$.

Applying (18) we obtain:

$$\lim_r \left\{ \sum_{t \in T} \lambda_t^r \begin{pmatrix} a_t - a \\ b_t - b \end{pmatrix} + \sum_{s \in S_+} \mu_s^r \begin{pmatrix} a_s - a \\ b_s - b \end{pmatrix} + \sum_{s \in S_-} \mu_s^r \begin{pmatrix} a_s + a \\ b_s + b \end{pmatrix} + \gamma_r \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} =$$

$$= \begin{pmatrix} a \\ b \end{pmatrix} - \lim_r \left\{ \sum_{t \in T} \lambda_t^r + \sum_{s \in S_+} \mu_s^r - \sum_{s \in S_-} \mu_s^r \right\} \begin{pmatrix} a \\ b \end{pmatrix} = 0_{n+1}$$

because $\sum_{t \in T} \lambda_t^r + \sum_{s \in S_+} \mu_s^r - \sum_{s \in S_-} \mu_s^r = \sum_{t \in T} \lambda_t^r + \sum_{s \in S} |\mu_s^r| = 1$, for all r.

Then, $0_{n+1} \in cl\left(\tilde{G}\right)$ where $\tilde{G}$ is associated to $\tilde{\sigma}$ in (19), and then (taking account $\tilde{G} \subset \tilde{H}$) either $0_{n+1} \in bd\left(\tilde{H}\right) \cap bd\left(\tilde{G}\right)$ or $0_{n+1} \in int\left(\tilde{H}\right)$. Any case, Theorem 4 entails $\tilde{\sigma} \in cl\left(\Omega_s\right)$.

Case 3. According to Theorem 4, the remaining case is $\sigma \in bd\left(\Omega_s\right)$, and the same theorem yields the aimed equality. ■

4.1 The finite case

There exists an increasing literature (see, e.g.[6], [9], [20], [21], [22] and [23]) about stability and well/ill-posedness for linearly constrained systems in the context of conic linear systems of the form

$$\delta : \quad \begin{array}{l} b - Ax \in C_Y, \\ x \in C_X, \end{array} \tag{20}$$

where $C_X \subset X$ and $C_Y \subset Y$ are closed convex cones in the finite-dimensional normed spaces X and Y. Here $b \in Y$ and $A \in L(X,Y)$ is a linear operator, with norm $\|A\| := \sup\{\|Ax\| \mid \|x\| \leq 1\}$. The parameter space of all the data instances (20) is endowed with the product norm

$$\|\delta\| = \|(A,b)\| := \max\{\|A\|, \|b\|\}. \tag{21}$$

This model includes our constraint system (1) when T is finite just by taking $C_X := \mathbb{R}^n$ and $C_Y := -\mathbb{R}_+^{|T|} \times \{0_m\}$, where $|T|$ denotes the cardinality of T (recall $|S| = m$).

Here we derive a finite-dimensional version of the distance to ill-posedness given by Theorem 9, following the steps from Theorems 1 and 2 in [9]. This approach requires as an additional hypothesis the consistency of the nominal system and, then, it actually provides the distance to inconsistency.

If we identify $a \in \mathbb{R}^n$ with the linear operator $x \mapsto a'x$, a suitable norm to be used in $\mathbb{R}^{n+1}$ is

$$\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| = \max\{\|a\|_*, |b|\}, \tag{22}$$

where the norm $\|\cdot\|$ in $X = \mathbb{R}^n$ is arbitrary. In $Y = \mathbb{R}^{T \cup S}$ we shall use the norm $\|\cdot\|_\infty$.

Specifically, our system $\sigma = \{a_t'x \geq b_t,\ t \in T;\ a_s'x = b_s,\ s \in S\}$, when T is finite, can be rewritten in the form (20), where the t-th row of the matrix A is a_t', and the t-th component of the vector b is b_t, for all $t \in T \cup S$. Then, using the norm (21), we obtain:

$$\begin{aligned} \|\sigma\| &= \max\{\|A\|, \|b\|\} = \max\left\{\max_{\|x\|\leq 1} \|Ax\|, \|b\|\right\} \\ &= \max_{t\in T} \max\{\|a_t\|_*, |b_t|\} = \max_{t\in T} \left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|. \end{aligned}$$

For the system δ in (20), [9] presents different mathematical programs each of whose optimal values provides either the exact distance to inconsistency, denoted by $\rho(\delta)$, or an approximation of $\rho(\delta)$ to within certain constants. In particular, when applied to σ, assumed consistent, Theorem 2 in [9] establishes that this distance coincides with the optimal value of the program

$$\begin{array}{ll} \mathrm{Inf}_{y,q,g} & \max\left\{\left\|A'y-q\right\|_*, \left|b'y+g\right|\right\} \\ \text{s.t.} & y\in C_Y^*,\ \left\|y\right\|_*=1,\ q\in C_X^*,\ g\geq 0, \end{array} \tag{23}$$

where $C_Y^* := \{y \in Y \mid y'z \geq 0, \text{ for all } z \in C_Y\}$.

Writing $(\lambda, \mu) := -y$, the program (23) is equivalent to

$$\begin{array}{ll} \mathrm{Inf}_{\lambda,\mu,g} & \max\left\{\left\|\sum_{t\in T}\lambda_t a_t + \sum_{s\in S}\mu_s a_s\right\|_*, \left|\sum_{t\in T}\lambda_t b_t + \sum_{s\in S}\mu_s b_s - g\right|\right\} \\ \text{s.t.} & \lambda\geq 0_T,\ \mu\in\mathbb{R}^S,\ \sum_{t\in T}\lambda_t + \sum_{s\in S}|\mu_s| = 1,\ g\geq 0. \end{array}$$

By defining $w_{n+1} := \sum_{t\in T}\lambda_t b_t + \sum_{s\in S}\mu_s b_s - g$, and according to (22), we obtain another equivalent program

$$\begin{array}{ll} \mathrm{Inf}_{\lambda,\mu} & \left\|\begin{pmatrix} w \\ w_{n+1}\end{pmatrix}\right\| \\ \text{s.t.} & \lambda\geq 0_T,\ \mu\in\mathbb{R}^S, \sum_{t\in T}\lambda_t + \sum_{s\in S}|\mu_s| = 1, \\ & w = \sum_{t\in T}\lambda_t a_t + \sum_{s\in S}\mu_s a_s,\ w_{n+1} \leq \sum_{t\in T}\lambda_t b_t + \sum_{s\in S}\mu_s b_s. \end{array}$$

In other words, we have

$$\rho(\sigma) = \inf\left\{\left\|\begin{pmatrix} w \\ w_{n+1}\end{pmatrix}\right\| \,\middle|\, \begin{pmatrix} w \\ w_{n+1}\end{pmatrix} \in G\right\} = d\left(0_{n+1}, G\right) = d\left(0_{n+1}, bd\left(G\right)\right).$$

The last equality comes from the fact that $0_{n+1} \notin int\,(G)$ because of the consistency of σ.

As a final remark, we point out that Theorem 9 extends the previous result in two different ways: T is arbitrary and σ is allowed to be inconsistent. Our approach also intends to emphasize the geometrical aspects underlying the stability of systems (1). To illustrate this point, let us come back to Example 2 (see Figure 1).

Example 3 (Example 2 revisited) Consider again the consistent system, in $\mathbb{R}^2$, $\sigma = \{x_1 + x_2 \geq 0,\ x_1 - x_2 \geq 0;\ x_2 = 0\}$. It is very easy to check that $d\left(0_3, bd\left(G\right)\right)$ is attained, with respect to $\|\cdot\|_\infty$, at $\left(\frac{1}{3}, \pm\frac{1}{3}, 0\right)'$. Now we attend to some details in the proof of Theorem 9. Taking $(a_1, a_2, b)' = \left(\frac{1}{3}, \frac{1}{3}, 0\right)'$ in (17) written, according to (18), as

$$\left(\tfrac{1}{3}, \tfrac{1}{3}, 0\right)' = 0\,(1,1,0)' + \tfrac{1}{3}\,(1,-1,0)' + \tfrac{2}{3}\,(0,1,0)',$$

system (19), one of those in which the distance to inconsistency is attained, reads as

$$\tilde{\sigma} = \left\{\tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 \geq 0,\ \tfrac{2}{3}x_1 - \tfrac{4}{3}x_2 \geq 0;\ \tfrac{-1}{3}x_1 + \tfrac{2}{3}x_2 = 0\right\}.$$

Note that the last two constraints of $\tilde{\sigma}$ are equivalent to $x_1 - 2x_2 \geq 0$ and $x_1 - 2x_2 = 0$. So, $\tilde{\sigma}$ is consistent but arbitrarily small perturbations can make it inconsistent. The reader can also see that $\tilde{\sigma}$ is ill-posed attending to $\tilde{G} = \tilde{H}$.

5 Metric regularity

As an application of the previous sections, we will extend [2, Corollary 3.4] to systems of linear equations and inequalities. The referred result establishes that the distance to inconsistency of an homogeneous linear semi-infinite inequality system, say $\theta = \{a_t'x \geq 0,\ t \in T\}$, is equal to the regularity modulus of an appropriate (set-valued) mapping at the origin.

Let $F : X \rightrightarrows Y$ be a set-valued mapping acting from a Banach space X to the subsets of a Banach space Y, and let $(\bar{x}, \bar{y}) \in gph\,(F) := \{(x, y) \in X \times Y \mid y \in F\,(x)\}$ (the graph of F). Then F is said to be *metrically regular at* $\bar{x}$ *for* $\bar{y}$ if there exists a constant $\kappa > 0$ such that

$$d\left(x, F^{-1}\,(y)\right) \leq \kappa d\,(y, F\,(x)) \text{ for all } (x, y) \text{ close to } (\bar{x}, \bar{y})\,, \tag{24}$$

where F^{-1} is the inverse of F, defined by $x \in F^{-1}\,(y) \Leftrightarrow y \in F\,(x)$.

The metric regularity is a basic quantitative property of mappings in variational analysis which is widely used in both theoretical and computational studies (see, e.g., [7] for further information about this concept). If we consider the generalized equation '$y \in F\,(x)$', the distance of the left hand side of (24) is the distance from the "approximate solution" x to the set $F^{-1}\,(y)$ of feasible solutions associated to the value y of the parameter. The distance of the right hand side of (24) is a kind of residual, usually much easier to compute or estimate than the left hand side. In particular, if we know an estimate for the rate of convergence of the residuals to zero, then we can evaluate the rate of convergence of a sequence of approximate solutions to an exact solution.

The infimum of κ for which (24) holds is the *regularity modulus*, denoted by $regF\,(\bar{x} \mid \bar{y})$ (defined as $+\infty$ if F is not metrically regular at $\bar{x}$ for $\bar{y}$). The *radius of metric regularity,* denoted by $radF\,(\bar{x} \mid \bar{y})$ is defined as the infimum of the "size" (Lipschitz modulus; see, e.g., [7]) of the perturbations needed to lose the metric regularity property. In [7] it is shown that $radF\,(\bar{x} \mid \bar{y}) \geq 1/regF\,(\bar{x} \mid \bar{y})$ (assuming that $gph\,(F)$ is locally closed at $(\bar{x}, \bar{y})$). The equality holds if either $dimX < \infty$ and $dimY < \infty$, or $gph\,(F)$ is a closed and convex cone and $(\bar{x}, \bar{y})$ is the origin in $X \times Y$. The paper [19] establishes the equality when X is an Asplund space and $dimY < \infty$, whereas [16] proves that the inequality is generically strict.

In the sequel we will consider the homogeneous system

$$\sigma_0 := \{a_t'x \geq 0,\ t \in T;\ a_s'x = 0,\ s \in S\} \tag{25}$$

in the continuous case; i.e., T is a compact Hausdorff space and $t \mapsto a_t$ is continuous on T (recall that $s \mapsto a_s$ is trivially continuous since we are considering the discrete topology in S). Let $\mathcal{C}\,(T, \mathbb{R})$ denote the space of functions $f : T \longrightarrow \mathbb{R}$ that are continuous on T, and consider the set-valued mapping $F : \mathbb{R}^n \longrightarrow \mathcal{C}\,(T, \mathbb{R}) \times \mathbb{R}^S \equiv \mathcal{C}\,(T \cup S, \mathbb{R})$ (endowed with the supremum norm) given by

$$F\,(x) := a\,(\cdot)'\,x - (\mathcal{C}\,(T, \mathbb{R}_+) \times \{0_S\})\,, \tag{26}$$

where 0_S is the zero function on S and $a(t) := a_t$ for all $t \in T \cup S$. So, a given $b \in \mathcal{C}(T,\mathbb{R}) \times \mathbb{R}^S$ belongs to $F(x)$ if and only if x is a feasible solution of (1). In other words, $F^{-1}(b)$ is the feasible set of (1). In this way, the metric regularity of F informs about the stability of (1) when $a(\cdot)$ remains fixed and $b(\cdot)$ is the parameter to be perturbed.

The radius of metric regularity of this mapping is given by

$$radF(\bar{x} \mid b) = \inf_{l \in L(\mathbb{R}^n, \mathcal{C}(T,\mathbb{R}) \times \mathbb{R}^S)} \left\{ \begin{array}{c} \|l\| \mid F + l \text{ not metrically regular} \\ \text{at } \bar{x} \text{ for } b + l(\bar{x}) \end{array} \right\},$$

where $L\left(\mathbb{R}^n, \mathcal{C}(T,\mathbb{R}) \times \mathbb{R}^S\right)$ denotes the set of all (continuous) linear functions from $\mathbb{R}^n$ to $\mathcal{C}(T,\mathbb{R}) \times \mathbb{R}^S$, endowed with the usual operator norm (see for instance §4.1). Specifically, if

$$l(x) = \begin{pmatrix} g(\cdot)' x \\ h(\cdot)' x \end{pmatrix} \text{ for } g \in \mathcal{C}(T,\mathbb{R}^n) \text{ and } h : S \longrightarrow \mathbb{R}^n,$$

we have

$$\|l\| = \max\left\{ \sup_{t \in T} \|g(t)\|_*, \ \max_{s \in S} \|h(s)\|_* \right\}.$$

From now on we will assume that the norm considered in $\mathbb{R}^{n+1}$ satisfies

$$\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| = \left\| \begin{pmatrix} a \\ -b \end{pmatrix} \right\| \text{ for all } \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{n+1}. \tag{27}$$

Note that any p-norm, but not any norm (see [25, Theorem 15.2]), verifies this condition. We shall consider $\mathbb{R}^n$ endowed with the norm

$$\|a\| := \left\| \begin{pmatrix} a \\ 0 \end{pmatrix} \right\| \text{ for all } a \in \mathbb{R}^n. \tag{28}$$

The reader can check that properties (27) and (28) also hold for the associated dual norms. Under these hypotheses we have the following result.

Theorem 10 [2, Theorem 3.1 and Corollary 3.2] *For any $(\bar{x}, b) \in gphF$ we have*

$$radF(\bar{x} \mid b) = \frac{1}{regF(\bar{x} \mid b)} = \inf\left\{ \|u\|_* \mid \begin{pmatrix} u \\ u'\bar{x} \end{pmatrix} \in E(b) \right\},$$

where $E(b)$ is the set defined in (8).

Now, the announced extension of [2, Corollary 3.4] to linear systems including equations reads as follows.

Proposition 3 *The distance to infeasibility of the system σ_0 given by* (25) *is equal to $radF(0_n \mid 0_{T \cup S})$, where F is given by* (26).

Proof. According to our previous comments we have $\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|_* = \left\| \begin{pmatrix} a \\ -b \end{pmatrix} \right\|_*$ for all $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{n+1}$, which entails $\left\| \begin{pmatrix} a \\ 0 \end{pmatrix} \right\|_* \leq \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|_*$. Moreover $\|a\|_* = \left\| \begin{pmatrix} a \\ 0 \end{pmatrix} \right\|_*$, and appealing to Theorem 9 we obtain

$$d\left(\sigma_0, bd\left(\Omega_c\right)\right) = d_{\|\cdot\|_*}\left(0_{n+1}, bd\left(G_0\right)\right) = \inf \left\{ \|a\|_* \mid \begin{pmatrix} a \\ 0 \end{pmatrix} \in E_0 \right\},$$

since for the homogeneous system σ_0 we have

$$G_0 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^n \mid \begin{pmatrix} u \\ 0 \end{pmatrix} \in E_0,\ v \leq 0 \right\}.$$

The proof finishes by just applying Theorem 10. ■

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