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Sufficient conditions for total ill-posedness in linear optimization*

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Abstract

This paper deals with the so-called total ill-posedness of linear optimization problems with an arbitrary (possibly infinite) number of constraints. We say that the nominal problem is totally ill-posed if it exhibits the highest unstability in the sense that arbitrarily small perturbations of the problem's coefficients may provide both, consistent (with feasible solutions) and inconsistent problems, as well as bounded (with finite optimal value) and unbounded problems, and also solvable (with optimal solutions) and unsolvable problems . In this paper we provide sufficient conditions for the total ill-posedness property exclusively in terms of the coefficients of the nominal problem.

Key words. Linear programming, semi-infinite programming, ill-posedness.

Mathematics Subject Classification (2000): 90C05, 65F22, 90C34, 15A39, 52A40.

1 Introduction

Consider the linear optimization problem in the Euclidean space, $\mathbb{R}^n$,

$$\pi: \quad \begin{array}{l} \text{Inf } c'x \\ \text{s.t. } a_t'x \geq b_t\,,\ t \in T, \end{array} \tag{1}$$

where c, x, $a_t \in \mathbb{R}^n$, $b_t \in \mathbb{R}$, and y' denotes the transpose of $y \in \mathbb{R}^n$. The non-empty *index set*, T, of the *constraint system*, $\sigma = \{a_t'x \geq b_t\,,\ t \in T\}$,

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is arbitrary (possibly infinite) and, so, the results of this paper hold, as a particular case, in ordinary linear programming (LP). When T is infinite the problem $\pi = (c, \sigma)$ is a *linear semi-infinite programming* (*LSIP*) *problem*. The *feasible set* of π is denoted by F, its *optimal value* by v, and the *optimal set* by F^{op}.

The *parameter space* of all the problems (1), with constraint systems having the same index set T, is denoted by Π. The different problems in Π, and their associated elements, are distinguished by means of sub(super)scripts. So, if π_1 also belongs to Π, we write $\pi_1 = \left(c^1, \sigma_1\right)$ and $\sigma_1 := \{\left(a_t^1\right)' x \geq b_t^1,$ $t \in T\}$, and its feasible set, optimal value and optimal set are, accordingly, denoted by F_1, v_1 and F_1^{op}, respectively.

Π_c denotes the subset of Π formed by all the *consistent problems* ($\pi \in \Pi_c \Leftrightarrow \sigma$ is consistent $\Leftrightarrow F \neq \emptyset \Leftrightarrow v < +\infty$), while $\Pi_i := \Pi \backslash \Pi_c$ represents the subset of all the *inconsistent problems*. Π_b denotes the subset of the *bounded problems*, that is those problems with finite optimal value (v finite). Finally, we denote by Π_s the subset of the *solvable problems*, that is those problems with non-empty optimal set ($F^{op} \neq \emptyset$). Obviously, $\Pi_s \subset \Pi_b \subset \Pi_c$.

We introduce an *extended distance* $\delta : \Pi \times \Pi \rightarrow [0, +\infty]$ by means of

$$\delta\left(\pi_1, \pi\right) := \max\left\{\left\|c^1 - c\right\|, d\left(\sigma_1, \sigma\right)\right\}, \tag{2}$$

where

$$d\left(\sigma_1, \sigma\right) := \sup_{t \in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|,$$

assuming that we have considered two arbitrary norms in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, denoted both by $\|\cdot\|$. In this way Π is endowed with the topology of the uniform convergence of the coefficients vectors [7, Chapter 10]. Given $\pi \in \Pi$ and $\widetilde{\Pi} \subset \Pi$, we write, as usual, $\delta(\pi, \widetilde{\Pi}) := \inf\{\delta\left(\pi, \widetilde{\pi}\right), \widetilde{\pi} \in \widetilde{\Pi}\}$, but now $\delta(\pi, \widetilde{\Pi})$ can take the value $+\infty$.

In (Π, δ), and also in the Euclidean space, $int(X)$, $cl(X)$, and $bd(X)$ denote the *interior set*, the *closure*, and the *boundary* of the set X, respectively. By $ext(X)$ we represent the *exterior* of X; i.e., the complementary set of $cl(X)$.

The stability of an optimization model is considered nowadays a crucial paradigmatic property, mainly by those scientists applying optimization to real world problems. Different authors (see, for instance, Todorov [13]) consider that an optimization problem is well-posed when it has a unique optimal solution and, moreover, this solution can be approximated by sequences of optimal solutions associated with any sequence of solvable problems converging to the nominal one. In [2] a general framework including other different notions of well-posedness in LSIP is developed. Some of these notions are closely related to the condition that a problem belongs to the interior of the set of solvable problems, $int(\Pi_s)$. (See, also, [1].)

Generically we may consider that a problem is ill-posed when arbitrarily small perturbations of its coefficients yield different kind of problems, namely, consistent/inconsistent, bounded/unbounded, solvable/unsolvable, etc. This idea leads us to say that the ill-posed problems, with respect to each one of these properties, are those problems belonging to the common boundary of the corresponding pair of sets; i.e., the sets $bd\,(\Pi_c)$, $bd\,(\Pi_b)$ and $bd\,(\Pi_s)$, respectively.

Concerning the ill-posedness with respect to the consistency, there are many contributions in the context of conic linear optimization ([6], [9], [10]), in the line of the outstanding paper of Renegar ([11]), whereas in [3] we have characterized this set of ill-posed problems, $bd\,(\Pi_c)$, in the LSIP setting. The influence of the distance to ill-posedness, $\delta(\pi, bd\,(\Pi_c))$, on the numerical complexity of certain algorithms is emphasized in [3], where the relationship of this distance with certain stability properties, as the Aubin property of the feasible set mapping (see, for instance, [12]), is also explored in detail.

Again in the LSIP setting, the set of ill-posed problems with respect to the solvability, $bd\,(\Pi_s)$, has been characterized in [4], and the problem of measuring the distance to this set, $\delta(\pi, bd\,(\Pi_s))$, is approached (either by means of an exact formula or through some lower/upper bounds) in [5].

Here in this paper we call *totally ill-posed* to those problems which are simultaneously unstable in all the previous senses, i.e., the problems belonging to the set

$$bd\,(\Pi_c) \cap bd\,(\Pi_b) \cap bd\,(\Pi_s)\,.$$

In [4, Thm. 1] we have proved that Π_b and Π_s have the same closure, the same interior and, therefore, the same boundary; i.e., $bd\,(\Pi_b) = bd\,(\Pi_s)$. This fact justifies the choice of

$$bd\,(\Pi_c) \cap bd\,(\Pi_s)$$

as a concept of *total ill-posedness* in linear optimization.

In [4, Thm. 3] we have also characterized the set $bd\,(\Pi_c) \cap bd\,(\Pi_s)$ in terms of two ingredients: the convex hull, in $\mathbb{R}^n$, of $\{a_t,\, t \in T;\, c\}$, denoted by Z^+, that is

$$Z^+ := conv\,(\{a_t,\, t \in T;\, c\})\,,$$

according to the notation introduced in §2, and the parameter set

$$cl\,(bd\,(\Pi_c) \cap \Pi_c)\,.$$

Since that this last set is not expressed in terms of the coefficients of π, the main objective of this paper is to derive some sufficient conditions to guarantee that π is totally ill-posed, in terms exclusively of the problem's coefficients. This is done in §3, whereas §4 presents different examples in order to show that these sufficient conditions are not necessary.

2 Preliminaries

This section gives account of the notation and basic definitions, results, and tools used later on. Given $\varnothing \neq X \subset \mathbb{R}^k$, by *conv*$(X)$ and *cone*(X) we denote the *convex hull* of X and the *conical convex hull* of X, respectively. It is assumed that *cone*$\,(X)$ always contains the zero-vector 0_k. We denote by X^o the (positive) *dual cone* of X given by

$$X^o := \left\{ y \in \mathbb{R}^k \mid y'x \geq 0 \text{ for all } x \in X \right\}.$$

If X is a closed convex set, $O^+(X)$ represents its *recession cone*, i.e.,

$$O^+(X) := \left\{ y \in \mathbb{R}^k \mid \text{for some } x \in X,\ x + \lambda y \in X \text{ for all } \lambda \geq 0 \right\}.$$

$\mathbb{R}_+^{(T)}$ denotes the cone of all the functions $\lambda : T \to \mathbb{R}_+$ taking positive values only at finitely many points of T.

If $\Lambda \subset \mathbb{R}$, we introduce the set

$$\Lambda X := \{\lambda x : \lambda \in \Lambda \text{ and } x \in X\}.$$

We denote by $\|.\|_*$ the *dual norm* of $\|.\|$, that is,

$$\|u\|_* := \max\left\{u'z \mid \|z\| \leq 1\right\}, \text{ for } u \in \mathbb{R}^k.$$

The following sets, associated with $\pi = (c, \sigma)$, are relevant in our analysis:

$$\begin{array}{ll}
A := conv\,(\{a_t,\ t \in T\})\,, & M := cone\,(\{a_t,\ t \in T\}) = \mathbb{R}_+ A, \\
C := conv\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T\right\}\right), & N := cone\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T\right\}\right) = \mathbb{R}_+ C, \\
H := C + \mathbb{R}_+ \begin{pmatrix} 0_n \\ -1 \end{pmatrix}, & K := N + \mathbb{R}_+ \left\{\begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\} = \mathbb{R}_+ H,
\end{array}$$

where $\mathbb{R}_+ := [0, +\infty[$. We say that H is the *hypographical set,* whereas M is called *first moment cone,* N is the *second moment cone,* and K is the *characteristic cone*.

If $\pi = (c, \sigma) \in \Pi_c$, with feasible set F, we say that $a'x \geq b$ is a *consequence* of σ if this inequality is satisfied at each point of F; i.e., $a'z \geq b$ for every $z \in F$.

The following proposition gathers different results which are applied throughout the paper.

Proposition 1 *Given* $\pi = (c, \sigma) \in \Pi$, *the following statements hold:*

(i) $\pi \in \Pi_c$ *if and only if*

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \notin cl\,(N)\,;$$

(ii) *If* $\pi \in \Pi_c$, $a'x \geq b$ *is a consequence of* σ *if and only if*

$$\begin{pmatrix} a \\ b \end{pmatrix} \in cl\,(K)\,; \tag{3}$$

(iii) *If* $\pi \in \Pi_c$, *then* $\pi \in int\,(\Pi_c)$ *if and only if* $0_{n+1} \notin cl\ (C)\,;$

(iv) *If* $\pi \in bd\,(\Pi_c) \cap \Pi_i$, *then* $0_n \in bd\,(A)\,.$

Proof. (i) constitutes a kind of extended *Gale theorem* (see, for instance, [7, Theorem 4.4]).

(ii) is the so-called (non-homogeneous) *Farkas Lemma* ([14]) and, as a consequence of this result, $cl\,(K)$ is also called the *consequent relations cone* of σ.

(iii) [8, Theorem 3.1].

(iv) [4, Lemma 1(ii)]. ■

The existence of infinitely many coefficient vectors when T is infinite gives rise to a pathological class of problems. This is the family

$$\Pi_\infty := \{\pi \in \Pi \mid \delta(\pi, bd(\Pi_c)) = +\infty\}\,.$$

The following proposition includes a characterization of these abnormal problems.

Proposition 2 *Given* $\pi \in \Pi$, *one has:*

(i) $\pi \in \Pi_\infty$ *if and only if* $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in O^+(cl(C))$;

(ii) $\Pi_\infty \subset \Pi_i$;

(iii) *If* $\pi \in \Pi_i \diagdown \Pi_\infty$, *then there exists* $\rho \geq 0$ *such that*

$$\begin{pmatrix} 0_n \\ \rho \end{pmatrix} \in cl\,(C)\,.$$

Proof. (i) [3, Proposition 1];

(ii) It is immediate consequence of the inclusion $O^+(cl(C)) \subset cl(N)$ and Proposition 1(i);

(iii) [3, Lemma 4]. ■

If T is finite, C is bounded and, therefore, $\Pi_\infty = \emptyset$ as a consequence of (i).

The following proposition provides a characterization of the ill-posedness with respect to the consistency, as well as a formula for the distance to ill-posedness. This formula translates the problem of measuring a distance in the parameter space Π into a problem of calculating an Euclidean distance.

Proposition 3 [3, Theorems 4, 5 and 6] *Let* $\pi \in \Pi \diagdown \Pi_\infty$. *Then, the following statements hold:*

(i) $\pi \in int\,(\Pi_i) \Leftrightarrow 0_{n+1} \in int\,(H)\,;$

(ii) $\pi \in int\,(\Pi_c) \Leftrightarrow 0_{n+1} \in ext\,(H)\,;$

(iii) $\pi \in bd\,(\Pi_c) \Leftrightarrow 0_{n+1} \in bd(H);$

(iv) $\delta(\pi, bd(\Pi_c)) = d\,(0_{n+1}, bd\,(H))\,.$

The following result, together with Proposition 3(iii), provides a characterization of totally ill-posedness in $\Pi \diagdown \Pi_\infty$. Note that if $\pi \in bd\,(\Pi_c)$ only the cases $\pi \in bd\,(\Pi_s)$ or $\pi \in ext\,(\Pi_s)$ are possible.

Theorem 1 *Let* $\pi \in bd\,(\Pi_c)\,.$ *Then:*

(i) $\pi \in bd\,(\Pi_s)$ *if and only if either* $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$ *or* $0_n \in bd\,(Z^+)\,;$

(ii) *If* $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)\,,$ *then* $0_{n+1} \in bd\,(C)\,.$ *The converse statement holds when* $\{b_t,\, t \in T\}$ *is bounded.*

Proof. (i) [4, Theorem 3];

(ii) [4, Theorem 4]. ■

Corollary 1 *Let* $\pi = (c, \sigma)$ *be an LP problem* (*T finite*). *Then,* π *is totally ill-posed if and only if* $0_{n+1} \in bd(H)$ *and either* $0_n \in bd\,(Z^+)$ *or* $0_{n+1} \in bd\,(C)$.

Proof. Apply Theorem 1 and Proposition 3(iii). ■

3 Sufficient conditions for $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$

Theorem 1(ii) states that, under the assumption of boundedness of the set $\{b_t, t \in T\}$, the condition $0_{n+1} \in bd\,(C)$ guarantees that a problem $\pi \in bd\,(\Pi_c)$ belongs also to $cl\,(bd\,(\Pi_c) \cap \Pi_c)$. This section, in fact, is devoted to establish new sufficient conditions for a problem $\pi \in bd\,(\Pi_c)$ to belong to $cl\,(bd\,(\Pi_c) \cap \Pi_c)$ (and then to $bd\,(\Pi_s)$, by virtue of Theorem 1(i)). The following subsets of $\mathbb{R}^{n+1}$, closely related to C, are key tools for this purpose.

Definition 1 *Let $\pi \in \Pi$ and $\mu \geq 0$, and define the following convex subsets of C :*

$$\widehat{C}_\mu := conv\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix} \middle| \left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \leq \mu,\, t \in T\right\}\right),$$

$$\tilde{C}_\mu := conv\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix} \middle| |b_t| \leq \mu,\, t \in T\right\}\right).$$

Theorem 2 *Let $\pi \in bd\,(\Pi_c)$. If there exists $\mu \geq 0$ such that $0_{n+1} \in bd\left(\widehat{C}_\mu\right)$, then $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$.*

Proof. Let $\pi \in bd\,(\Pi_c)$, and suppose the nontrivial case $\pi \in \Pi_i$. Suppose also that there exists $\mu \geq 0$ such that $0_{n+1} \in bd\left(\widehat{C}_\mu\right)$.

By Proposition 2(iii), there exists $\rho \geq 0$ such that $\begin{pmatrix} 0_n \\ \rho \end{pmatrix} \in cl\,(C)$, and let

$$\rho_0 := \sup\left\{\rho \geq 0 \mid \begin{pmatrix} 0_n \\ \rho \end{pmatrix} \in cl\,(C)\right\}.$$

It must be $\rho_0 < +\infty$ because, otherwise, $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in O^+(cl(C))$, entailing $\pi \in \Pi_\infty$ (by Proposition 2(i)), which contradicts $\pi \in bd\,(\Pi_c)$.

Take $\mu_0 \geq \max\{\mu, \rho_0 + 1\}$. Since $\begin{pmatrix} 0_n \\ \mu_0 \end{pmatrix} \notin cl\,(C)$, we can strictly separate $\begin{pmatrix} 0_n \\ \mu_0 \end{pmatrix}$ and $cl\,(C)$; i.e., there exist $\begin{pmatrix} v \\ \alpha \end{pmatrix} \in \mathbb{R}^{n+1} \setminus \{0_{n+1}\}$ and $\beta \in \mathbb{R}$ such that $\begin{pmatrix} v \\ \alpha \end{pmatrix}' \begin{pmatrix} 0_n \\ \mu_0 \end{pmatrix} < \beta$ and $\begin{pmatrix} v \\ \alpha \end{pmatrix}' y \geq \beta$ for all $y \in cl\,(C)$. In particular, since

$$0_{n+1} \in bd\left(\widehat{C}_\mu\right) \subset cl\left(\widehat{C}_\mu\right) \subset cl\left(\widehat{C}_{\mu_0}\right) \subset cl\,(C),$$

one has $\beta \leq 0$ and, consequently, $\alpha < 0$ (since $\mu_0 > 0$). Moreover, $\alpha\mu_0 < \beta$ and $\begin{pmatrix} v \\ \alpha \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} \geq \beta$ for all $t \in T$; then $\begin{pmatrix} \frac{v}{-\alpha} \\ -1 \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} \geq \frac{\beta}{-\alpha} > -\mu_0$.

On the other hand, since $\pi \in bd\,(\Pi_c) \cap \Pi_i$, Proposition 1(iv) ensures that $0_n \in bd\,(A)$ and, therefore, we can take a supporting halfspace to A at 0_n; i.e., there exists a non-zero vector $u \in \mathbb{R}^n$ such that $a_t' u \geq 0$ for all $t \in T$.

Take $w := \frac{v}{-\alpha}$ and write

$$u^r := u + \frac{1}{r} w, \;\; r = 1, 2, ...,$$

($u^r = u$, for all r, if $w = 0_n$). Define the problem $\pi_r := (c, \sigma_r)$ where the coefficient vectors of σ_r are given, for each $t \in T$, by

$$\begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} := \begin{cases} \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \left[\frac{1}{r} b_t - a_t' u^r\right]_+ \begin{pmatrix} u^r / \left\| u^r \right\|_2^2 \\ 0 \end{pmatrix} & \text{, if } \left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| > \mu_0, \\ \begin{pmatrix} a_t \\ b_t \end{pmatrix} - \frac{1}{r} \left(a_t' w - b_t \right) \begin{pmatrix} u^r / \left\| u^r \right\|_2^2 \\ 0 \end{pmatrix} & \text{, if } \left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \leq \mu_0, \end{cases}$$

where $[\alpha]_+ := \max\{0, \alpha\}$.

First let us see that $\{\pi_r\}$ converges to π. We know that $\begin{pmatrix} w \\ -1 \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} > -\mu_0$, that is, $a_t' w > b_t - \mu_0$ for all $t \in T$. So, and because $a_t' u \geq 0$ for all $t \in T$,

$$a_t' u^r - \frac{1}{r} b_t = a_t' u + \frac{1}{r} a_t' w - \frac{1}{r} b_t \geq \frac{-1}{r} \mu_0,$$

that is, $\frac{1}{r} b_t - a_t' u^r \leq \frac{\mu_0}{r}$ with $\frac{\mu_0}{r} \geq 0$, and therefore $\left[\frac{1}{r} b_t - a_t' u^r\right]_+ \leq \frac{\mu_0}{r}$. So, if $\left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| > \mu_0$ then

$$\begin{aligned} \left\| \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| &= \left[\frac{1}{r} b_t - a_t' u^r\right]_+ \left\| \begin{pmatrix} u^r / \left\| u^r \right\|_2^2 \\ 0 \end{pmatrix} \right\| \\ &\leq \frac{\mu_0}{r} \left\| \begin{pmatrix} u^r / \left\| u^r \right\|_2^2 \\ 0 \end{pmatrix} \right\|. \end{aligned} \quad (4)$$

In the case that $\left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \leq \mu_0$ we have

$$\begin{aligned} \left\| \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| &= \left\| -\frac{1}{r} \left(a_t' w - b_t \right) \begin{pmatrix} u^r / \left\| u^r \right\|_2^2 \\ 0 \end{pmatrix} \right\| \\ &\leq \frac{\mu_0}{r} \left\| \begin{pmatrix} w \\ -1 \end{pmatrix} \right\|_* \left\| \begin{pmatrix} u^r / \left\| u^r \right\|_2^2 \\ 0 \end{pmatrix} \right\|. \end{aligned} \tag{5}$$

Thus, since $\{u^r\}$ converges to u, the sequence $\left\{ \left\| \begin{pmatrix} u^r / \left\| u^r \right\|_2^2 \\ 0 \end{pmatrix} \right\| \right\}$ converges to $\left\| \begin{pmatrix} u / \left\| u \right\|_2^2 \\ 0 \end{pmatrix} \right\|$ and, hence, it is bounded. Thus, by (4) and (5) we have that $d\left(\sigma_r, \sigma\right) \leq \frac{k}{r}$ for certain constant $k \geq 0$. Then $\{\sigma_r\}$ converges to σ and, consequently, $\{\pi_r\}$ converges to π.

Now let us see that $\pi_r \in \Pi_c$; in fact, we are going to see that $ru^r \in F_r$ for all $r \in \mathbb{N}$. If $\left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| > \mu_0$ then

$$\begin{aligned} &\left(\begin{pmatrix} a_t \\ b_t \end{pmatrix} + \left[\tfrac{1}{r} b_t - a_t' u^r \right]_+ \begin{pmatrix} u^r / \left\| u^r \right\|_2^2 \\ 0 \end{pmatrix} \right)' \begin{pmatrix} r u^r \\ -1 \end{pmatrix} \\ &\quad = r \left(a_t' u^r - \tfrac{1}{r} b_t + \left[\tfrac{1}{r} b_t - a_t' u^r \right]_+ \right) = r \left[a_t' u^r - \tfrac{1}{r} b_t \right]_+ \geq 0, \end{aligned}$$

and, in the case that $\left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \leq \mu_0$, we have

$$\begin{aligned} &\left(\begin{pmatrix} a_t \\ b_t \end{pmatrix} - \tfrac{1}{r} \left(a_t' w - b_t \right) \begin{pmatrix} u^r / \left\| u^r \right\|_2^2 \\ 0 \end{pmatrix} \right)' \begin{pmatrix} r u^r \\ -1 \end{pmatrix} \\ &\quad = r \left(a_t' \left(u + \tfrac{1}{r} w \right) - \tfrac{1}{r} b_t - \tfrac{1}{r} \left(a_t' w - b_t \right) \right) = r a_t' u \geq 0. \end{aligned}$$

Finally let us see that $0_{n+1} \in cl\left(C_r\right)$, which entails (applying Proposition 1(iii)) that $\pi_r \in bd\left(\Pi_c\right)$. In fact, since $0_{n+1} \in cl\left(\widehat{C}_{\mu_0}\right)$, we can write $0_{n+1} = \lim_k \sum_{t \in T} \lambda_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix}$ with $\left\{\lambda^k\right\} \subset \mathbb{R}_+^{(T)}$, $\sum_{t \in T} \lambda_t^k = 1$, and $\lambda_t^k = 0$ if $\left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| > \mu_0$, for each k. As a consequence of that we get

$$\begin{aligned} 0_{n+1} &= \lim_k \sum_{t\in T} \frac{-1}{r}\lambda_t^k \left(\begin{pmatrix} a_t \\ b_t \end{pmatrix}' \begin{pmatrix} w \\ -1 \end{pmatrix} \right) \begin{pmatrix} u^r / \|u^r\|_2^2 \\ 0 \end{pmatrix} \\ &= \lim_k \sum_{t\in T} \lambda_t^k \frac{-1}{r} \left(a_t' w - b_t \right) \begin{pmatrix} u^r / \|u^r\|_2^2 \\ 0 \end{pmatrix} \\ &= \lim_k \sum_{t\in T} \lambda_t^k \left\{ \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\}, \end{aligned}$$

and, therefore,

$$0_{n+1} = \lim_k \sum_{t\in T} \lambda_t^k \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} \in cl\,(C_r)\,.$$

■

The converse statement in the previous theorem is not true in general as Example 1 shows. Moreover, this example, together with Example 2, motivates the following new sufficient condition for $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$ in terms of the set $\tilde{C}_\mu$.

Theorem 3 *Let $\pi \in bd\,(\Pi_c)$. Suppose that there exists $u \in M^o (= O^+(F))$ and $\mu > 0$ satisfying $\begin{pmatrix} u \\ -1 \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} \geq -\mu$, for all $t \in T$, and such that $0_{n+1} \in bd\left(\tilde{C}_\mu\right)$. Then $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$.*

Proof. Under the current hypothesis, let us consider the sequence of problems $\pi_r := (c, \sigma_r)$, $r = 1, 2, ...$, where σ_r is the system whose coefficient vectors are, for each $t \in T$,

$$\begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} := \begin{cases} \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \left[\frac{1}{r} b_t - a_t' u\right]_+ \begin{pmatrix} u / \|u\|_2^2 \\ 0 \end{pmatrix}, & \text{if } |b_t| > \mu, \\ \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \frac{1}{r} b_t \begin{pmatrix} u / \|u\|_2^2 \\ 0 \end{pmatrix}, & \text{if } |b_t| \leq \mu, \end{cases}$$

We verify first that $\{\pi_r\}$ converges to π. Indeed, since $\begin{pmatrix} u \\ -1 \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} \geq -\mu$ then, for all $t \in T$ and $r \in \mathbb{N}$,

$$\frac{1}{r} b_t - a_t' u = \left(\frac{1}{r} - 1 \right) a_t' u - \frac{1}{r} \left(a_t' u - b_t \right) \leq \frac{\mu}{r},$$

where we have used that $a_t' u \geq 0$ (because $u \in M^o$). So we easily check that

$$d(\sigma_r, \sigma) \leq \frac{\mu}{r} \left\| \begin{pmatrix} u/\|u\|_2^2 \\ 0 \end{pmatrix} \right\| \underset{r\to\infty}{\longrightarrow} 0,$$

and $\{\sigma_r\}$ converges to σ and, consequently, $\{\pi_r\}$ converges to π.

Now let us see that $\pi_r \in \Pi_c$ by proving that $ru \in F_r$. In fact, if $|b_t| > \mu$ we have

$$\begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix}' \begin{pmatrix} ru \\ -1 \end{pmatrix} = r\left(a_t' u - \frac{1}{r} b_t + \left[\frac{1}{r} b_t - a_t' u \right]_+ \right) = r\left[a_t' u - \frac{1}{r} b_t \right]_+ \geq 0.$$

If $|b_t| \leq \mu$ then

$$\begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix}' \begin{pmatrix} ru \\ -1 \end{pmatrix} = r\left(a_t' u - \frac{1}{r} b_t + \frac{1}{r} b_t \right) = r a_t' u \geq 0.$$

Finally, let us see that $\pi_r \in bd(\Pi_c)$. Since $0_{n+1} \in bd\left(\tilde{C}_\mu\right) \subset cl\left(\tilde{C}_\mu\right)$, there exists a sequence $\left\{\lambda^k\right\} \subset \mathbb{R}_+^{(T)}$, with $\sum_{t\in T} \lambda_t^k = 1$, and $\lambda_t^k = 0$ if $|b_t| > \mu$, for each $k \in \mathbb{N}$, such that $0_{n+1} = \lim_k \sum_{t\in T} \lambda_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix}$, from which we obtain $0 = \lim_k \sum_{t\in T} \lambda_t^k b_t$. Multiplying both members by $\frac{1}{r} \begin{pmatrix} u/\|u\|_2^2 \\ 0 \end{pmatrix}$ one obtains $0_{n+1} = \lim_k \sum_{t\in T} \lambda_t^k \frac{1}{r} b_t \begin{pmatrix} u/\|u\|_2^2 \\ 0 \end{pmatrix}$ and, therefore,

$$0_{n+1} = \lim_k \sum_{t\in T} \lambda_t^k \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} \in cl(C_r).$$

Then $\pi_r \in bd(\Pi_c)$ (again by virtue of Proposition 1(iii)). ■

Example 3, in the following section, shows that the converse statement in the previous theorem is not true in general.

4 Examples and counterexamples

The following example shows that the converse statement in Theorem 2 is not true in general.

Example 1 Consider the problem

$$\begin{array}{lll} \pi : & \text{Inf} & -x_1 + 0x_2 \\ & \text{s.t.} & 0x_1 + 0x_2 \geq 1, \text{ if } t = 0, \\ & & 0x_1 - x_2 \geq 0, \text{ if } t = 1, \\ & & x_1 + rx_2 \geq 0, \text{ if } t = 2r - 1, \ r = 2, 3, \ldots, \\ & & rx_1 + 0x_2 \geq r, \text{ if } t = 2r, \ r = 1, 2, 3, \ldots . \end{array}$$

Graphically one has the situation represented in Fig. 1, where the last block of constraints has not been represented.

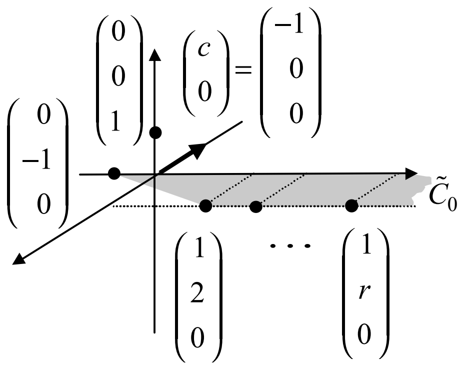

Fig. 1. The converse of Thm. 2 is not true in general.

It is easy to check, as we can appreciate in Fig. 1, that $\pi \in bd\,(\Pi_c)$ (because $0_3 \in bd(H)$) and $0_3 \notin bd\left(\widehat{C}_\mu\right)$ whichever $\mu \geq 0$ we take. Let us see however that $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$. With this aim, consider for each $s \in \mathbb{N}$ the problem π_s which comes from changing in π the constraint corresponding to $t = 0$ by $\frac{1}{s}x_1 + 0x_2 \geq 1$. It is easy to check that $\pi_s \in bd\,(\Pi_c) \cap \Pi_c$ for all $s \in \mathbb{N}$ (in fact $(s, 0)' \in F_s$) and $\delta\,(\pi, \pi_s) = \dfrac{1}{s} \underset{s \to \infty}{\longrightarrow} 0$. Therefore one concludes that $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$.

In view of the previous example, one could think that, if $0_{n+1} \in bd\left(\tilde{C}_\mu\right)$ for some $\mu \geq 0$ (in the previous example $0_3 \in bd\left(\tilde{C}_0\right)$), it must be $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$. However, the next example shows that it is necessary to require some additional hypothesis, as we have done in Theorem 3.

Example 2 Consider the problem

$$\begin{array}{lll} \pi: & \text{Inf} & -x_1+0x_2 \\ & \text{s.t.} & 0x_1-x_2\geq 0, \text{if } t=0, \\ & & x_1+tx_2\geq 0,\ \text{if } t=2k-1,\ k\in\mathbb{N}, \\ & & x_1+tx_2\geq t,\ \text{if } t=2k,\ k\in\mathbb{N}, \end{array}$$

whose coefficient vectors are represented in Fig. 2

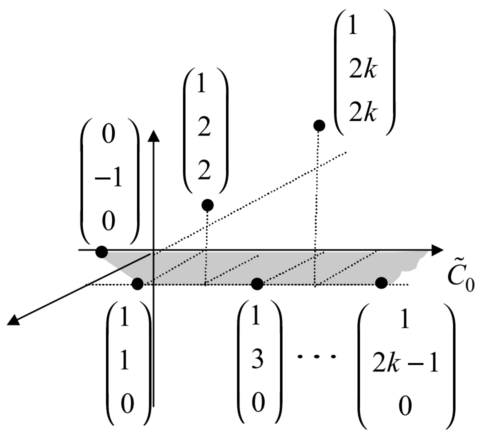

Fig. 2. $0_3\in bd\left(\tilde{C}_\mu\right)$ for all $\mu\geq 0$ but $\pi\notin cl\left(bd\left(\Pi_c\right)\cap\Pi_c\right).$

For all $\mu\geq 0$ one has $0_3\notin bd\left(\widehat{C}_\mu\right)$ but $0_3\in bd\left(\tilde{C}_\mu\right)$. We will see that $\pi\notin cl\left(bd\left(\Pi_c\right)\cap\Pi_c\right)$. It is easy to check that $(0,1,1)'$ is a recession direction of $cl\left(C_1\right)$ for all $\pi_1\in\Pi$ with $\delta\left(\pi_1,\pi\right)<+\infty$. Let $\pi_1\in\Pi$ with $\delta\left(\pi_1,\pi\right)\leq\varepsilon<\frac{1}{8}$ and write

$$\begin{pmatrix}a_0^1\\b_0^1\end{pmatrix}=\begin{pmatrix}\varepsilon_{01}\\-1+\varepsilon_{02}\\\varepsilon_{03}\end{pmatrix},\ \begin{pmatrix}a_1^1\\b_1^1\end{pmatrix}=\begin{pmatrix}1+\varepsilon_{11}\\1+\varepsilon_{12}\\\varepsilon_{13}\end{pmatrix}.$$

Let us check that $\pi_1\in int\left(\Pi_c\right)$ if $\varepsilon_{01}>0$ and $\pi_1\in\Pi_i$ if $\varepsilon_{01}\leq 0$, from which we will obtain $\pi\notin cl\left(bd\left(\Pi_c\right)\cap\Pi_c\right)$. Indeed, if $\varepsilon_{01}>0$ one has $a_{t1}^1\geq\varepsilon_{01}$ for all $t\in\mathbb{N}\cup\{0\}$ (where a_{t1}^1 represents the first coordinate of a_t^1), then $H_1\subset[\varepsilon_{01},+\infty[\times\mathbb{R}^2$ and, therefore, $0_3\in ext\left(H_1\right)$, that is, $\pi_1\in int\left(\Pi_c\right)$. Suppose now that $\varepsilon_{01}\leq 0$. Then

$$\frac{1+\varepsilon_{11}}{1+\varepsilon_{11}-\varepsilon_{01}}\begin{pmatrix}a_0^1\\b_0^1\end{pmatrix}+\frac{-\varepsilon_{01}}{1+\varepsilon_{11}-\varepsilon_{01}}\begin{pmatrix}a_1^1\\b_1^1\end{pmatrix}=(0,\alpha,\beta)'\in C_1$$

and, due to the fact that $\varepsilon < \frac{1}{8}$, we have $|\beta| < -\alpha$. Thus, taking into account that $(0,1,1)'$ is a recession direction of $cl\,(C_1)$,

$$(0,0,\beta-\alpha)' = (0,\alpha,\beta)' + (-\alpha)\,(0,1,1)' \in cl\,(C_1),$$

and then $(0,0,1)' \in cl\,(N_1)$; that is, $\pi_1 \in \Pi_i$ (by Proposition 1(i)).

The following example shows that the converse statement in Theorem 3 is not true in general.

Example 3 Consider the problem, in $\mathbb{R}$,

$$\begin{aligned} \pi: \;\; & \text{Inf} \; -x \\ & \text{s.t.} \;\; 0x \geq 1, \text{if } t=0, \\ & \qquad \tfrac{1}{t}x \geq -t, \text{ if } t = 2k,\; k \in \mathbb{N}, \\ & \qquad x \geq -t, \text{ if } t = 2k-1,\; k \in \mathbb{N}. \end{aligned}$$

If we take $u \in M^o = \mathbb{R}_+$, one observes that $\begin{pmatrix} u \\ -1 \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} \geq -\mu$ is satisfied if and only if $\mu \geq 1$. Then we will see that $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$ despite that $0_2 \notin bd\left(\tilde{C}_\mu\right)$ for every $\mu \geq 1$. Graphically we observe what happens in Fig. 3:

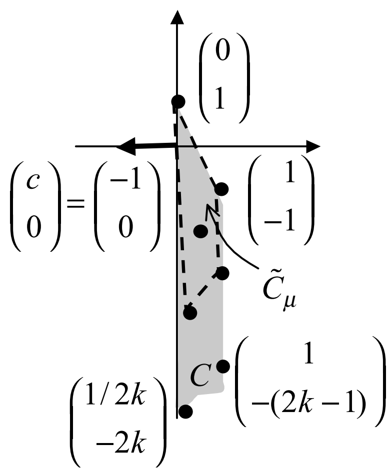

Fig. 3. $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$ but $0_2 \notin bd\left(\tilde{C}_\mu\right)$ for all $\mu \geq 1$.

For proving that $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$, take for each $r \in \mathbb{N}$ the problem

$$\begin{array}{lll} \pi_r: & \text{Inf} & -x \\ & \text{s.t.} & 0x \geq 1, \quad \text{if } t = 0 \\ & & \frac{1}{t}x \geq -t, \quad \text{if } t = 2k,\ k \neq r,\ k \in \mathbb{N}, \\ & & 0x \geq -t, \quad \text{if } t = 2r, \\ & & x \geq -t, \quad \text{if } t = 2k-1,\ k \in \mathbb{N}. \end{array}$$

For each $r \in \mathbb{N}$ one has $\pi_r \in bd\,(\Pi_c)$ and $0_2 \in bd\left(\widehat{C}_r\right)_{2r}$ (where $\left(\widehat{C}_r\right)_{2r}$ denotes the "$\widehat{C}_\mu$" corresponding to σ_r with $\mu = 2r$). Then $\pi_r \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$ by virtue of Theorem 2. Moreover $\delta\,(\pi_r, \pi) = \frac{1}{2r} \underset{r\to\infty}{\longrightarrow} 0$ and, therefore, $\pi \in cl\,(bd\,(\Pi_c) \cap \Pi_c)$.

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