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M. J. Cánovas, A. L. Dontchev, M. A. López and J. Parra

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Centro de Investigación Operativa
Universidad Miguel Hernández de Elche
Avda. de la Universidad s/n
03202 Elche (Alicante)
cio@umh.es

TRABAJOS I+D

METRIC REGULARITY OF SEMI-INFINITE CONSTRAINT SYSTEMS[1]

M. J. Cánovas, A. L. Dontchev, M. A. López and J. Parra[2]

Dedicated to R. T. Rockafellar on his 70th Birthday

Abstract. We obtain a formula for the modulus of metric regularity of a mapping defined by a semi-infinite system of equalities and inequalities. Based on this formula, we prove a theorem of Eckart-Young type for such set-valued infinite-dimensional mappings: given a metrically regular mapping F of this kind, the infimum of the norm of a linear function g such that $F + g$ is not metrically regular is equal to the reciprocal to the modulus of regularity of F. The Lyusternik-Graves theorem gives a straightforward extension of these results to nonlinear systems. We also discuss the distance to infeasibility for semi-infinite linear inequality systems.

Key Words. Semi-infinite programming, metric regularity, distance to inconsistency, conditioning.

AMS Subject Classification. 90C34, 49J53, 49K40, 65Y20.

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[2]M. J. Cánovas and J. Parra are with Operations Research Center, Miguel Hernández University of Elche, 03202 Elche (Alicante), Spain, A. L. Dontchev is with Mathematical Reviews, Ann Arbor, MI 48107-8604, and M. A. López is with Department of Statistics and Operations Research, University of Alicante, 03071 Alicante, Spain.

1 Introduction

This paper is about regularity of mappings defined by semi-infinite constraint systems of inequalities and equalities. We find a formula for the modulus of metric regularity, prove an Eckart-Young-type theorem, and provide an expression for the distance to infeasibility of semi-infinite inequality systems.

Let $F : X \rightrightarrows Y$ be a set-valued mapping acting from a Banach space X to the subsets of a Banach space Y and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Here $\operatorname{gph} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ is the graph of F. Then F is said to be *metrically regular at* $\bar{x}$ *for* $\bar{y}$ if there exists a constant $\kappa > 0$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \text{ for all } (x, y) \text{ close to } (\bar{x}, \bar{y})\,. \tag{1.1}$$

We denote by $d(x, C)$ the distance from a point x to a set C, that is, $d(x, C) = \inf_{y \in C} \|x - y\|$, and by F^{-1} the inverse of F, that is, $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$.

The metric regularity is a basic quantitative property of mappings in variational analysis which is widely used in both theoretical and computational studies. For an illustration of how this concept works, let $\bar{x}$ be a solution of the inclusion $\bar{y} \in F(x)$ for some given $\bar{y}$, let F be metrically regular at $\bar{x}$ for $\bar{y}$, and let x_{a} and y_{a} be approximations to $\bar{x}$ and $\bar{y}$, respectively, so that $(x_{\mathrm{a}}, y_{\mathrm{a}})$ is sufficiently close to $(\bar{x}, \bar{y})$. Then, from (1.1), the distance from x_{a} to the set of solutions $F^{-1}(y_{\mathrm{a}})$ is bounded by the constant κ times the residual $d(y_{\mathrm{a}}, F(x_{\mathrm{a}}))$. Usually, the residual is easy to compute or estimate, while finding a solution might be considerably difficult. Under mild conditions, the metric regularity of the mapping F guarantees that there exists a solution to the inclusion $y_{\mathrm{a}} \in F(x)$ at distance from x_{a} proportional to the residual. In particular, if we know an estimate for the rate of convergence of the residual to zero, then we can evaluate the rate of convergence of a sequence of approximate solutions to an exact solution.

The infimum of κ for which (1.1) holds is the *regularity modulus*, denoted by $\operatorname{reg} F(\bar{x}|\bar{y})$; the case when F is not metrically regular at $\bar{x}$ for $\bar{y}$ corresponds to $\operatorname{reg} F(\bar{x}|\bar{y}) = \infty$.

The concept of metric regularity has its roots in the Banach open mapping principle: a linear and bounded operator $A : X \to Y$, denoted $A \in L(X, Y)$, is metrically regular (at any point in its graph) if and only if A maps X onto Y. The regularity modulus of an $A \in L(X, Y)$ satisfies

$$\operatorname{reg} A = \sup_{y \in B_Y} d(0, A^{-1}(y)),$$

where B_Y denotes the closed unit ball in the space Y. If in addition A^{-1} is single-valued, then $\operatorname{reg} A = \|A^{-1}\|$. For set-valued mappings $F : X \rightrightarrows Y$ with closed convex graph, the metric regularity is characterized through the classical Robinson-Ursescu theorem:

Theorem 1.1. (Robinson-Ursescu) *A mapping $F : X \rightrightarrows Y$ with closed and convex graph is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if $\bar{y}$ is in the interior of the range of F.*

In this paper we evaluate the regularity modulus of mappings defined by semi-infinite constraint systems. We start with the following simple example which will be used throughout the paper:

Example 1.2. For fixed parameters $u \in I\!R^n$ and $v \in I\!R$ consider the set-valued mapping

$$I\!R^n \ni x \mapsto F_{\binom{u}{v}}(x) := u^{\mathsf{T}} x - v - I\!R_+,$$

where $^{\mathsf{T}}$ denotes the transposition (the elements of $I\!R^n$ are regarded as column vectors) and $I\!R_+ := [0, \infty)$. Then, in particular, $F^{-1}_{\binom{u}{v}}(0)$ is the set of solutions of the single inequality $u^{\mathsf{T}} x - v \geq 0$. Consider any norm $\|\cdot\|$ in $I\!R^n$ and let $\|\cdot\|_*$ denote the dual norm, i.e. $\|u\|_* := \sup \left\{ u^{\mathsf{T}} x \mid \|x\| \leq 1 \right\}$. For $u \neq 0$ we have

$$d(x, F^{-1}_{\binom{u}{v}}(y)) = \frac{[y - (u^{\mathsf{T}} x - v)]_+}{\|u\|_*},$$

where the distance corresponds to the norm $\|\cdot\|$ and $[a]_+$ denotes the positive part of a. Also, we have

$$d(y, F_{\binom{u}{v}}(x)) = \left[y - (u^{\mathsf{T}} x - v) \right]_+ .$$

Then, directly from the definition (1.1) we obtain

$$\text{(1.2)} \qquad \operatorname{reg} F_{\binom{u}{v}}(\bar{x}\,|\,0) = \begin{cases} 0 & \text{if } \ u^{\mathsf{T}} \bar{x} > v, \\ 1/\|u\|_* & \text{if } \ u^{\mathsf{T}} \overline{x} = v \ \text{ and } \ u \neq 0, \\ \infty & \text{if } \ u^{\mathsf{T}} \bar{x} = v \ \text{ and } \ u = 0. \end{cases}$$

□

There is a fast growing literature on metric regularity in variational analysis; recent overviews to the topic can be found in [4], [11] and [24]. In this paper we closely follow the notation and terminology of [24].

The metric regularity of a mapping F is known to be equivalent to two other properties: the openness of linear rate of F and the Aubin property of the inverse F^{-1}. The metric regularity of F at $\bar{x}$ for $\bar{y}$ implies that for any neighborhood O of $\bar{x}$, $F^{-1}(y) \cap O \neq \emptyset$ for all y sufficiently close to $\bar{y}$. Also, if $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) < \infty$, then $\operatorname{reg} F(x\,|\,y) < \infty$ for all $(x, y) \in \operatorname{gph} F$ close to $(\bar{x}, \bar{y})$.

For a function $f : X \to Y$ and a point $\bar{x} \in \operatorname{int} \operatorname{dom} f$, we denote by $\operatorname{lip} f(\bar{x})$ the Lipschitz modulus of f at $\bar{x}$,

$$\operatorname{lip} f(\bar{x}) = \limsup_{x, x' \to \bar{x},\ x \neq x'} \frac{\|f(x) - f(x')\|}{\|x - x'\|}.$$

Also, recall that a function $g : X \to Y$ is strictly differentiable at $\bar{x} \in \operatorname{int} \operatorname{dom} g$ with a strict derivative mapping $Dg(\bar{x}) \in L(X, Y)$ if

$$\operatorname{lip}(g - Dg(\bar{x}))(\bar{x}) = 0.$$

A central result in the theory of metric regularity is a theorem which goes back to Lyusternik and Graves, for more see, e.g., [3] and [11], which we will use here in the following form:

Theorem 1.3 (Lyusternik-Graves)**.** *Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ at which $\operatorname{gph} F$ is locally closed. Let $g : X \to Y$ be a function which is strictly differentiable at $\bar{x}$ with strict derivative $Dg(\bar{x})$. Then*

$$\operatorname{reg}(F+g)(\bar{x}\,|\,\bar{y}+g(\bar{x})) = \operatorname{reg}(F(\cdot) + Dg(\bar{x})(\cdot - \bar{x}))(\bar{x}\,|\,\bar{y}).$$

The modern variants of the Lyusternik-Graves theorem give not only the invariance of the metric regularity of a mapping F with respect to linearization of its smooth part, but also a bound for the smallest perturbation g for which the mapping $F + g$ is not metrically regular. Obtaining an exact expression for the "radius" of metric regularity, as well as for other well-posedness properties, is related to *conditioning* of mappings and goes back to a theorem by Eckart and Young [6] for matrices. In optimization, the Eckart-Young theorem was extended in the framework of feasibility of constraint systems in the pioneering work of Renegar [21]. Related theoretical results with applications to various optimization problems have been recently obtained in a number of papers, see e.g. [2], [19], [20], [25], [26], and [27]. A general estimate for the "distance to non-regularity" of set-valued mappings was established in [4], as follows:

Theorem 1.4 (radius theorem for metric regularity)**.** *Consider a mapping $F : X \rightrightarrows Y$ and any $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ at which $\operatorname{gph} F$ is locally closed. Then*

$$\inf_{g:X\to Y}\left\{ \operatorname{lip} g(\bar{x}) \,\middle|\, F+g \text{ not metrically regular at } \bar{x} \text{ for } \bar{y}+g(\bar{x}) \right\} \geq \frac{1}{\operatorname{reg} F(\bar{x}\,|\,\bar{y})}. \tag{1.4}$$

In addition, if either $\dim X < \infty$ and $\dim Y < \infty$ or $\operatorname{gph} F$ is a closed and convex cone and $(\bar{x}, \bar{y})$ is the origin in $X \times Y$, then

$$\inf_{g:X\to Y}\left\{ \operatorname{lip} g(\bar{x}) \,\middle|\, F+g \text{ not metrically regular at } \bar{x} \text{ for } \bar{y}+g(\bar{x}) \right\} = \frac{1}{\operatorname{reg} F(\bar{x}\,|\,\bar{y})}. \tag{1.5}$$

Moreover, the infimum is unchanged if taken with respect to affine mappings g of rank one and then $\operatorname{lip} g(\bar{x})$ is equal to the norm of the linear part of g.

As an application of this result, a characterization to the distance to infeasibility, as defined by Renegar [21], was obtained in [4] via *homogenization* of the original mapping. Mordukhovich [18] subsequently proved the equality (1.5) for mappings acting from an Asplund space to a finite-dimensional space. Recently, Ioffe showed that in infinite-dimensions the inequality in (1.4) is generally strict [12], but equality (1.5) still holds [13] when F is single-valued and the perturbation is nonlinear. Results of this type for other related regularity properties of mappings, including subregularity and strong regularity, are presented in [5].

In this paper we consider specific set-valued mappings acting from a finite-dimensional space to subsets of the Banach space of continuous functions over a compact Hausdorff space, which describe constraint systems in semi-infinite programming. The graphs of such mappings are closed convex cones, hence, from Theorem 1.4, the radius equality (1.5) holds for them at the origin. By using the specifics of these mappings and following the path of the previous work of Cánovas et al. [1], we derive a formula for their regularity moduli at any point in the graph and prove a corresponding equality (1.5) with linear perturbations g. This result sharpens the inequality in (1.4) as equality of the form (1.5) for basic models in optimization involving mappings acting in infinite-dimensional spaces.

We consider a semi-infinite system in $I\!R^n$ of the form

$$\text{(1.6)} \qquad \left\{ \begin{array}{ll} a(t)^{\mathsf{T}}x \geq b(t) & \text{for all } t \in T, \\ p(s)^{\mathsf{T}}x = q(s) & \text{for all } s \in S, \end{array} \right.$$

where T is a compact, possibly infinite, Hausdorff space, S is a finite set with cardinality m, where $m \leq n$, and such that $S \cap T = \emptyset$, the functions $a : T \to I\!R^n$ and $b : T \to I\!R$ are continuous on T, and $p : S \to I\!R^n$, $q : S \to I\!R$ are regarded as functions over a discrete domain.

Let $\mathcal{C}(T, I\!R)$ denote the space of functions $f : T \to I\!R$ that are continuous on T. The system (1.6) can be described with the help of the following set-valued mapping $\mathbf{F}$ acting from $I\!R^n$ to the subsets of $\mathcal{C}(T, I\!R) \times I\!R^S$:

$$\text{(1.7)} \qquad I\!R^n \ni x \mapsto \mathbf{F}(x) = \begin{pmatrix} a(\cdot)^{\mathsf{T}}x \\ p(\cdot)^{\mathsf{T}}x \end{pmatrix} - \begin{pmatrix} \mathcal{C}(T, I\!R_+) \\ 0_S \end{pmatrix},$$

where $\mathcal{C}(T, I\!R_+)$ is the set of continuous functions on T whose values are nonnegative scalars and 0_S is the zero function on S. Then $x \in I\!R^n$ and a pair $(b, q) \in \mathcal{C}(T, I\!R) \times I\!R^S$ satisfy $(b, q) \in \mathbf{F}(x)$ if and only if x is a solution of the system (1.6). Equivalently, $\mathbf{F}^{-1}(b, q)$ is the set of feasible points of (1.6). Here and further the functions a and p are fixed. The space $\mathcal{C}(T, I\!R) \times I\!R^S$ is equipped with the product norm

$$\|(b, q)\| := \max\left\{\|b\|_\infty, \|q\|_\infty\right\},$$

where $\|\cdot\|_\infty$ is used to denote the supremum norms for both $\mathcal{C}(T, I\!R)$ and $I\!R^S$.

First, observe that $\operatorname{gph}\mathbf{F}$ is a closed convex cone and hence, by the Robinson-Ursescu Theorem (Theorem 1.1), the metric regularity of $\mathbf{F}$ at $\bar{x}$ for $(b, q) \in \mathbf{F}(\bar{x})$ is equivalent to the condition $(b, q) \in \operatorname{int}\operatorname{rge}\mathbf{F}$, that is, (1.6) is consistent for any (b', q') close to (b, q). Next, note that the mapping $\mathbf{F}$ in (1.7) can be defined also as

$$I\!R^n \ni x \mapsto \mathbf{F}(x) = \left\{ (b, q) \in \mathcal{C}(T, I\!R) \times I\!R^S \;\middle|\; \begin{array}{ll} b(t) \in a(t)^{\mathsf{T}}x - I\!R_+ & \text{for } \; t \in T \\ q(s) = p(s)^{\mathsf{T}}x & \text{for } \; s \in S \end{array} \right\}.$$

To give a "pointwise representation" of this mapping we define the set $\tilde{T} := T \cup S$ and introduce the mapping

$$\text{(1.8)} \qquad \tilde{T} \times I\!R^n \ni (v, x) \mapsto F(v, x) = \left\{ w \in I\!R \;\middle|\; \begin{array}{ll} w \in a(v)^{\mathsf{T}}x - I\!R_+ & \text{for } \; v \in T, \\ w = p(v)^{\mathsf{T}}x & \text{for } \; v \in S. \end{array} \right\}.$$

Next, we study the relation between the regularity moduli of the mappings $\mathbf{F}$ and F.

To put this question in a more general perspective, let X and Y be Banach spaces with Y being reflexive, let $\bar{T}$ be a topological space, and let $F : \bar{T} \times X \rightrightarrows Y$ be a set-valued mapping. Define an associated set-valued mapping $\mathcal{F} : X \rightrightarrows \mathcal{C}(\bar{T}, Y)$ in the following way

$$X \ni x \mapsto \mathcal{F}(x) = \left\{ y \in \mathcal{C}(\bar{T}, Y) \,\middle|\, y(t) \in F(t, x) \text{ for } t \in \bar{T} \right\}. \tag{1.9}$$

We will next show that, under certain hypotheses that hold in our context, the regularity modulus of $\mathcal{F}$ is bounded from below by the supremum of the regularity moduli of $F(t, \cdot)$ for $t \in \bar{T}$. Specifically, we have the following result:

Theorem 1.5. *Let $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{F}$, where $\mathcal{F}$ is the set-valued mapping in (1.9) for a mapping F, and assume the following:*

(a) *There is a neighborhood O of $\bar{x}$ such that for every $(t, x) \in \bar{T} \times O$ the set $F(t, x)$ is nonempty, closed and convex;*

(b) *For each $(t_0, x) \in \bar{T} \times O$ and for each $u \in F(t_0, x)$ there exists a function $y \in \mathcal{C}(\bar{T}, Y)$ such that $y(t_0) = u$ and $y(t) \in F(t, x)$ for all $t \in \bar{T}$;*

(c) *$\mathcal{F}$ is lower semicontinuous at $(\bar{x}, \bar{y})$, meaning that for each sequence $\{x^r\} \subset X$ converging to $\bar{x}$, there exists a sequence $\{y^r\}$ converging to $\bar{y}$ and such that $y^r \in \mathcal{F}(x^r)$, for all $r = 1, 2, \cdots$.*

Then,

$$\operatorname{reg} \mathcal{F}(\bar{x} \,|\, \bar{y}) \geq \sup_{t \in \bar{T}} \operatorname{reg} F(t; \bar{x} \,|\, \bar{y}(t)), \tag{1.10}$$

where, for any given $t \in \bar{T}$, by $\operatorname{reg} F(t; \bar{x} \,|\, \bar{y}(t))$ we denote the regularity modulus of $F(t, \cdot)$ at $\bar{x}$ for $\bar{y}(t)$.

Proof. On the contrary, assume that there is a positive γ such that

$$\operatorname{reg} \mathcal{F}(\bar{x} \,|\, \bar{y}) < \gamma < \sup_{t \in \bar{T}} \operatorname{reg} F(t; \bar{x} \,|\, \bar{y}(t)).$$

Then for some $t_0 \in \bar{T}$, $\gamma < \operatorname{reg} F(t_0; \bar{x} \,|\, \bar{y}(t_0))$. Consequently, there exist sequences $x^r \to \bar{x}$ and $u^r \to \bar{y}(t_0)$ such that

$$d(x^r, F(t_0, \cdot)^{-1}(u^r)) > \gamma d(u^r, F(t_0, x^r)) \text{ for } r = 1, 2, \cdots, \tag{1.11}$$

where $F(t_0, \cdot)^{-1}$ is the inverse of $F(t_0, \cdot)$. Without loss of generality, let $x^r \in O$ for all r. Since Y is reflexive and the values of F are closed convex sets (condition (a)), for each r there exists $\bar{u}^r \in F(t_0, x^r)$ satisfying

$$\|\bar{u}^r - u^r\| = d(u^r, F(t_0, x^r)). \tag{1.12}$$

From the assumption (c) there exists a sequence of continuous functions z^r converging uniformly to $\bar{y}$ and with $z^r \in \mathcal{F}(x^r)$ for all r. Since $z^r(t_0) \in F(t_0, x^r)$ for any r, from (1.12) we have

$$\lim_{r \to \infty} \|\bar{u}^r - u^r\| \leq \lim_{r \to \infty} \|z^r(t_0) - u^r\| = 0.$$

Hence, $\bar{u}^r \to \bar{y}(t_0)$ as $r \to \infty$. Then, by the continuous selection assumption (b), for every r we can find a function $\tilde{y}^r \in \mathcal{C}(\bar{T}, Y)$ such that

$$\tilde{y}^r(t_0) = \bar{u}^r \quad \text{and} \quad \tilde{y}^r(t) \in F(t, x^r) \quad \text{for all } \ t \in \bar{T}.$$

Making use of Urysohn's lemma, for each $r = 1, 2, \cdots$, we construct a continuous function $\theta^r : \bar{T} \to [0, 1]$ satisfying

$$\theta^r(t) := \begin{cases} 1 & \text{if } \ \|\tilde{y}^r(t) - z^r(t)\| \leq \|\tilde{y}^r(t_0) - z^r(t_0)\|, \\ 0 & \text{if } \ \|\tilde{y}^r(t) - z^r(t)\| \geq \|\tilde{y}^r(t_0) - z^r(t_0)\| + \frac{1}{r}. \end{cases}$$

For $r = 1, 2, \cdots$ define the functions

$$\hat{z}^r(t) = \theta^r(t)\,\tilde{y}^r(t) + (1 - \theta^r(t))\,z^r(t) \quad t \in \bar{T},$$

Since $\theta^r(t_0) = 1$, for all r we have

$$\hat{z}^r(t_0) = \tilde{y}^r(t_0) = \bar{u}^r.$$

Noting that $\lim_{r\to\infty} \|\bar{u}^r - z^r(t_0)\| = 0$ and that $z^r \to \bar{y}$ uniformly in $\bar{T}$, the estimate

$$\begin{aligned} \|\hat{z}^r(t) - \bar{y}(t)\| &\leq \theta^r(t)\|\tilde{y}^r(t) - z^r(t)\| + \|\, z^r(t) - \bar{y}(t)\| \\ &< \|\tilde{y}^r(t_0) - z^r(t_0)\| + \frac{1}{r} + \|\, z^r(t) - \bar{y}(t)\| \\ &= \|\bar{u}^r - z^r(t_0)\| + \frac{1}{r} + \|\, z^r(t) - \bar{y}(t)\| \quad \text{for all } \ t \in \bar{T} \end{aligned}$$

yields that the sequence $\hat{z}^r$ converges uniformly in $\bar{T}$ to $\bar{y}$ as $r \to \infty$. Since $F(t, x^r)$ is convex for any $t \in \bar{T}$ and r (condition (a)) we conclude that $\hat{z}^r \in \mathcal{F}(x^r)$.

Consider now the sequence of continuous functions

$$\hat{y}^r(t) := \hat{z}^r(t) - (\bar{u}^r - u^r), \quad t \in \bar{T}, \quad r = 1, 2, \cdots.$$

Since $\lim_{r\to\infty} \|\bar{u}^r - u^r\| = 0$, the sequence $\hat{y}^r$ is uniformly convergent to $\bar{y}$. Since $\hat{y}^r(t_0) = u^r$ and $\mathcal{F}^{-1}(\hat{y}^r) \subset F(t_0, \cdot)^{-1}(u^r)$, we have

$$d(x^r, F(t_0, \cdot)^{-1}(u^r)) \leq d(x^r, \mathcal{F}^{-1}(\hat{y}^r)). \tag{1.13}$$

Moreover, since $d(\hat{y}^r, \hat{z}^r) = \|\bar{u}^r - u^r\|$, we obtain $d(\hat{y}^r, \mathcal{F}(x^r)) \leq \|\bar{u}^r - u^r\|$. This inequality combined with (1.11), (1.12) and (1.13) gives us

$$d(\hat{y}^r, \mathcal{F}(x^r)) < \frac{1}{\gamma} d(x^r, F(t_0, \cdot)^{-1}(u^r)) \leq \frac{1}{\gamma} d(x^r, \mathcal{F}^{-1}(\hat{y}^r)) \ \text{ for all } \ r = 1, 2, \cdots.$$

This contradicts the assumption $\operatorname{reg} \mathcal{F}(\bar{x}\,|\,\bar{y}) < \gamma$. ◻

Observe that the conditions (a), (b) and (c) in Theorem 1.5 hold for the mappings $\mathbf{F}$ in (1.7) and the corresponding mapping F in (1.8) for the linear semi-infinite inequality system. Hence, from (1.10) we obtain

$$\operatorname{reg}\mathbf{F}(\bar{x}\,|\,\bar{y}) \geq \sup_{v\in\tilde{T}} \operatorname{reg} F(v;\bar{x}\,|\,\bar{y}(v)).$$

The following two examples show that this inequality is strict, in general. In the first example the left hand side is infinite while the right hand side is finite. In the second example both sides are finite but different.

Example 1.6. Consider the scalar semi-infinite inequality

$$tx \geq t^2 - 1 \quad \text{for all } t \in [-1,1].$$

Let $\bar{y}(t) = t^2 - 1$, $t \in [-1,1]$. The associated mapping defined in (1.8) is

$$F(t,x) := tx - I\!R_+$$

and then, from (1.9),

$$\mathcal{F}(x) := \{y \in \mathcal{C}([-1,1], I\!R) \mid tx \geq y(t) \text{ for all } t \in [-1,1]\}.$$

From the formula (1.2) we have

$$\operatorname{reg} F(t;0\,|\,\bar{y}(t)) = \begin{cases} 0 & \text{if } t \in (-1,1), \\ 1 & \text{if } t = -1 \text{ or } 1. \end{cases}$$

However, $\operatorname{reg}\mathcal{F}(0\,|\,\bar{y}) = \infty$. Indeed, let $y^r(t) = t^2 - 1 + \frac{1}{r}$, for all $t \in [-1,1]$ and observe that for a sufficiently large r, the function y^r will be in any $\mathcal{C}([0,1], I\!R)$ neighborhood of $\bar{y}$. On the other hand, the set

$$\mathcal{F}^{-1}(y^r) = \left\{x \in I\!R \mid tx \geq t^2 - 1 + \frac{1}{r} \text{ for all } t \in [-1,1]\right\}$$

is empty for any r. Hence, $\mathcal{F}$ is not metrically regular at 0 for $\bar{y}$. □

Example 1.7. Consider the semi-infinite inequality

$$(1-t)x_1 + (-1+2t)x_2 \geq t^2 - t \quad \text{for all} \quad t \in [0,1],$$

for which

$$F(t,x) = (1-t)x_1 + (-1+2t)x_2 - I\!R_+.$$

Let $\bar{y}(t) = t^2 - t$, $t \in [0,1]$. Assuming that $I\!R^2$ is endowed with the Euclidean norm and using (1.2) we obtain

$$\operatorname{reg} F(t;0_2\,|\,\bar{y}(t)) = \begin{cases} 0, & \text{if } t \in (0,1), \\ \frac{1}{\sqrt{2}}, & \text{if } t = 0, \\ 1 & \text{if } t = 1, \end{cases}$$

where 0_2 is the origin in $\mathbb{R}^2$. To obtain a lower bound for $\operatorname{reg} \mathcal{F}(0_2 \,|\, \bar{y})$ for the associated mapping $\mathcal{F}$ defined in (1.9) we consider the sequence of functions $y^r(t) = t^2 - t + \frac{1}{r}$, $t \in [0,1]$, $r \in \mathbb{R}$. This sequence converges to $\bar{y}$ in $\mathcal{C}([0,1], \mathbb{R})$ as $r \to \infty$. We have

$$\mathcal{F}^{-1}(y^r) = \left\{ x \in \mathbb{R}^2 \mid x_1 - x_2 \geq \frac{1}{r},\, x_2 \geq \frac{1}{r} \right\}.$$

The inequality $(1-t)x_1 + (-1+2t)x_2 \geq \frac{1}{r} + t^2 - t$, for any $t \in (0,1)$, is always satisfied when both inequalities $x_1 - x_2 \geq \frac{1}{r}$ and $x_2 \geq \frac{1}{r}$ hold, and then elementary calculation gives us

$$d(0_2, \mathcal{F}^{-1}(y^r)) = \frac{\sqrt{5}}{r}\,.$$

Furthermore,

$$\mathcal{F}(0_2) = \left\{ y \in \mathcal{C}([0,1], \mathbb{R}) \mid y(t) \leq 0, \text{ for } t \in [0,1] \right\}$$

and hence

$$d(y^r, \mathcal{F}(0_2)) = \frac{1}{r} \quad \text{for all } r.$$

Thus, $\operatorname{reg} \mathcal{F}(0_2 \,|\, \bar{y}) \geq \sqrt{5}$. From the formula for the regularity modulus derived in the following section we will see that in this example $\operatorname{reg} \mathcal{F}(0_2 \,|\, \bar{y})$ is actually equal to $\sqrt{5}$. $\square$

In the following Section 2 we derive a formula for the modulus of metric regularity of the semi-infinite system (1.6). In view of Proposition 1.5 and the examples above, the main observation is that in order to evaluate the regularity modulus one has to consider not only the active constraints at the reference point but also their convex combinations. Then in Section 3 we prove a radius theorem for the metric regularity of the associated mapping (1.7) of the form of the equality (1.5), and apply this result to derive a formula for the distance to infeasibility for the semi-infinite system with inequality constraints only. In Section 4 we give an extension of the result obtained for a nonlinear semi-infinite constraint system by using the Lyusternik-Graves theorem (Theorem 1.3).

2 Metric regularity of linear semi-infinite systems

Consider the linear semi-infinite system (1.6),

$$\begin{cases} a(t)^{\mathsf{T}} x \geq b(t) & \text{for all } t \in T, \\ p(s)^{\mathsf{T}} x = q(s) & \text{for all } s \in S, \end{cases}$$

and the associated mapping $\mathbf{F}$ in (1.7). Recall that the *Slater constraint qualification* (SCQ) condition holds when the vectors $p(s)$, $s \in S$, are linearly independent and there exists a point

$\hat{x} \in I\!R^n$ satisfying $a(t)^\mathsf{T}\hat{x} > b(t), t \in T, p(s)^\mathsf{T}\hat{x} = q(s), s \in S$. Also, recall the *extended Mangasarian-Fromovitz constraint qualification* (EMFCQ) condition: denoting the set of indices associated with active inequalities at $\bar{x} \in \mathbf{F}^{-1}(b, q)$ by

$$T(\bar{x}) := \{t \in T \mid a(t)^\mathsf{T}\bar{x} = b(t)\},$$

we say that the EMFCQ condition is satisfied at $(\bar{x}, (b, q))$ when the vectors $p(s),\ s \in S$, are linearly independent and either $T(\bar{x}) = \emptyset$ or the following system has a solution

$$\begin{cases} a(t)^\mathsf{T}y > 0 & \text{for all } t \in T(\bar{x}), \\ p(s)^\mathsf{T}y = 0 & \text{for all } s \in S. \end{cases}$$

In [10], p. 311, the SCQ condition is called the *strong* Slater constraint qualification. The EMFCQ condition was introduced in the semi-infinite context in [14] based on a condition given in [9] (see [15] for a parametric version). The relationship between these conditions, the Robinson constraint qualification [22] and the metric regularity of the feasible set mapping in parametric nonlinear semi-infinite programming is explored in detail in [16].

The following set plays a central role in our analysis:

$$E(b, q) := \left\{ \sum_{t \in T} \lambda(t) \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} + \sum_{s \in S} \mu(s) \begin{pmatrix} p(s) \\ q(s) \end{pmatrix} \;\middle|\; \begin{array}{l} \lambda \in I\!R_+^{(T)},\ \mu \in I\!R^S \\ \sum_{t \in T} \lambda(t) + \sum_{s \in S} |\mu(s)| = 1 \end{array} \right\}, \tag{2.1}$$

where $I\!R_+^{(T)}$ is the set of all functions $\lambda : T \to I\!R_+$ with *finite support*, that is, taking nonnegative values only at finitely many points in T.

In the following theorem we give a summary of various properties that are equivalent to the metric regularity of the mapping $\mathbf{F}$ considered. We apply results from [16] but also exhibit new conditions which are used in the following sections. Theorem 2.1 is an adapted to the present setting version of Theorem 6.1 in [8]; for completeness, we give a condensed proof with a few shortcuts. Here and further, for a given set A we denote by $\operatorname{co} A$, $\operatorname{span} A$ and $\operatorname{cone} A$ the convex hull of A, the linear subspace generated by A and the conical convex hull of A, respectively.

Theorem 2.1. *For $(b, q) \in \operatorname{rge} \mathbf{F}$ the following are equivalent:*

(i) *For any $\bar{x} \in \mathbf{F}^{-1}(b, q)$ the mapping $\mathbf{F}$ is metrically regular at $\bar{x}$ for (b, q);*

(ii) $0_{n+1} \notin E(b, q)$;

(iii) $0_{n+1} \notin \operatorname{co}\left\{ \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, t \in T \right\} + \operatorname{span}\left\{ \begin{pmatrix} p(s) \\ q(s) \end{pmatrix}, s \in S \right\}$ *and the vectors $p(s), s \in S$, are linearly independent;*

(iv) *The system (1.6) satisfies the EMFCQ condition at any $\bar{x} \in \mathbf{F}^{-1}(b, q)$;*

(v) *The system (1.6) satisfies the SCQ condition;*

(vi) *$\mathbf{F}^{-1}$ is lower semicontinuous at (b, q).*

Proof. *(i)⇒(ii)* : According to the Robinson-Ursescu theorem, $(b, q) \in \operatorname{int} \operatorname{rge} \mathbf{F}$ and we can take $\varepsilon > 0$ such that $\|(b, q) - (b_1, q_1)\| < \varepsilon$ implies $(b_1, q_1) \in \operatorname{rge} \mathbf{F}$. Reasoning by contradiction, assume the existence of $\lambda = (\lambda(t))_{t\in T} \in I\!R_+^{(T)}$ and $\alpha(s) \in I\!R$, $s \in S$, such that $\sum_{t\in T} \lambda(t) + \sum_{s\in S} |\alpha(s)| = 1$ and

$$0_{n+1} = \sum_{t\in T} \lambda(t) \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} + \sum_{s\in S} \alpha(s) \begin{pmatrix} p(s) \\ q(s) \end{pmatrix}.$$

Let us consider (b_1, q_1) defined as follows:

$$b_1(t) = b(t) + \frac{\varepsilon}{2},\ t \in T, \quad q_1(s) = q(s) + \operatorname{sign} \alpha(s) \frac{\varepsilon}{2},\ s \in S.$$

Then we have $\|(b, q) - (b_1, q_1)\| = \varepsilon/2$ and

$$\sum_{t\in T} \lambda(t) \begin{pmatrix} a(t) \\ b_1(t) \end{pmatrix} + \sum_{s\in S} \alpha(s) \begin{pmatrix} p(s) \\ q_1(s) \end{pmatrix} = \frac{\varepsilon}{2} \begin{pmatrix} 0_n \\ 1 \end{pmatrix}.$$

The generalized Gale alternative theorem ([8], Corollary 3.1.1) leads to the contradiction $(b_1, q_1) \notin \operatorname{rge} \mathbf{F}$.

$(ii) \Rightarrow (iii)$: First, we prove that

$$0_{n+1} \notin \operatorname{co} \left\{ \begin{pmatrix} a(t) \\ b(t) \end{pmatrix},\ t \in T \right\} + \operatorname{span} \left\{ \begin{pmatrix} p(s) \\ q(s) \end{pmatrix},\ s \in S \right\}.$$

Reasoning by contradiction, assume that

$$0_{n+1} = \sum_{t\in T} \lambda(t) \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} + \sum_{s\in S} \alpha(s) \begin{pmatrix} p(s) \\ q(s) \end{pmatrix},$$

for certain $\lambda = (\lambda(t))_{t\in T} \in I\!R_+^{(T)}$ with $\sum_{t\in T} \lambda(t) = 1$ and $\alpha(s) \in I\!R$, $s \in S$. Dividing in the latter equality by $\sum_{t\in T} \lambda(t) + \sum_{s\in S} |\alpha(s)|$ we conclude that $0_{n+1} \in E(b, q)$, thus contradicting (ii).

Now, let us prove that the vectors $p(s)$, $s \in S$, are linearly independent. Otherwise we could write

$$0_n = \sum_{s\in S} \alpha(s) p(s) \text{ with } \alpha := \sum_{s\in S} |\alpha(s)| > 0.$$

Then, multiplying by any $\bar{x} \in \mathbf{F}^{-1}(b, q)$, we have

$$0 = \sum_{s\in S} \frac{\alpha(s)}{\alpha} p(s)^\mathsf{T} \bar{x} = \sum_{s\in S} \frac{\alpha(s)}{\alpha} q(s)$$

and therefore

$$0_{n+1} = \sum_{s\in S} \frac{\alpha(s)}{\alpha} \begin{pmatrix} p(s) \\ q(s) \end{pmatrix} \in E(b, q),$$

contradicting (ii).

$(iii)\Rightarrow(iv)$: Given $\bar{x} \in \mathbf{F}^{-1}(b,q)$ consider the non-trivial case $T(\bar{x}) \neq \emptyset$. We will first show that

$$0_n \notin \operatorname{co}\{a(t),\ t \in T(\bar{x})\} + \operatorname{span}\{p(s),\ s \in S\}.$$

On the contrary, assume that there exists $\lambda \in I\!R_+^{(T(\bar{x}))}$ such that $\sum_{t \in T(\bar{x})} \lambda(t) = 1$ and $\alpha(s) \in I\!R$, $s \in S$, satisfying

$$0_n = \sum_{t \in T(\bar{x})} \lambda(t) a(t) + \sum_{s \in S} \alpha(s) p(s).$$

Multiplying this equality by $\bar{x}$ one obtains

$$0 = \sum_{t \in T(\bar{x})} \lambda(t) b(t) + \sum_{s \in S} \alpha(s) q(s),$$

which together with the preceding equation leads to contradiction with *(iii)*. Now, the so-called Generalized Motzkin Alternative Theorem ([8], Theorem 3.5) applies, to conclude that the system

$$\{a(t)^\top y > 0,\ t \in T(\bar{x});\ p(s)^\top y = 0,\ s \in S\}$$

has a solution and hence the EMFCQ condition holds at $\bar{x}$.

$(iv)\Rightarrow(v)$: Pick a feasible point $\bar{x} \in \mathbf{F}^{-1}(b,q)$. If $T(\bar{x}) = \emptyset$, the point $\bar{x}$ satisfies the requirement of the SCQ condition. Let $T(\bar{x}) \neq \emptyset$ and take a solution $\bar{y}$ of the system

$$\{a(t)^\top y > 0,\ t \in T(\bar{x});\ p(s)^\top y = 0,\ s \in S\}.$$

If we replace the inequality constraints by the unique convex constraint $f(x) \leq 0$ with

$$f(x) := \sup_{t \in T}\{b(t) - a(t)^\top x\},$$

we then can apply an argument similar to the one in the proof of in the implication $(ii)\Rightarrow(i)$ of Theorem 7.2 in [8]. Specifically, Valadier's formula (see e.g. Theorem A.7 in [8]) yields the following expression for the subdifferential of f at $\bar{x}$:

$$\partial f(\bar{x}) = \operatorname{co}\{-a(t), t \in T(\bar{x})\},$$

and the directional derivative $f'(\bar{x};\bar{y})$ satisfies

$$f'(\bar{x};\bar{y}) = \sup\left\{u^\top \bar{y} \mid u \in \partial f(\bar{x})\right\} = \sup\left\{-a(t)^\top \bar{y} \mid t \in T(\bar{x})\right\} < 0,$$

taking into account the compactness of $T(\bar{x})$. Thus, if $\lambda > 0$ is small enough, one has $f(\bar{x}+\lambda\bar{y}) < 0$, which means that $\bar{x} + \lambda\bar{y}$ is a Slater point of (1.6).

$(v) \Rightarrow (vi)$: Choose an open set $U \subset I\!R^n$ such that $U \cap \mathbf{F}^{-1}(b,q) \neq \emptyset$; we will now prove the existence of $\varepsilon > 0$ such that $\|(b,q)-(b',q')\| < \varepsilon$ implies $U \cap \mathbf{F}^{-1}(b',q') \neq \emptyset$. To do so, take $\tilde{x} \in U \cap \mathbf{F}^{-1}(b,q)$ and any Slater vector $\hat{x}$ for (1.6), and observe that any point in the form $x(\lambda) := \lambda\hat{x} + (1-\lambda)\tilde{x}$, $\lambda \in (0,1]$ satisfies the SCQ condition. Let $\lambda_0 \in (0,1]$ be such that $x(\lambda_0) \in U$. Because of the continuity of $a(\cdot)$ and $b(\cdot)$ and the compactness of T, there exists $\rho > 0$ such that $a(t)^\mathsf{T} x(\lambda_0) \geq b(t) + \rho$ for all $t \in T$. Hence, there also exists $\varepsilon_1 > 0$ such that the combination of $\|x - x(\lambda_0)\| < \varepsilon_1$ and $\|b - b'\|_\infty < \varepsilon_1$ implies

$$x \in U \quad \text{and} \quad a(t)^\mathsf{T} x \geq b'(t) \quad \text{for all} \quad t \in T. \tag{2.2}$$

In the case that m (the cardinality of S) is less than n, since $p(s), s \in S$, are linearly independent, we can add vectors $p^{m+1}, \cdots, p^n$ such that $\{p(s),\ s \in S;\ p^{m+1}, \cdots, p^n\}$ is a basis of $I\!R^n$. Define $q^{m+j} := (p^{m+j})^\mathsf{T} x(\lambda_0)$, $j = 1, \cdots, n-m$. Then the system

$$\begin{cases} p(s)^\mathsf{T} x = q'(s), & s \in S, \\ (p^{m+j})^\mathsf{T} x = q^{m+j}, & j = 1, \cdots, n-m, \end{cases}$$

has a unique solution x' which depends continuously on q'. This implies the existence of $\varepsilon_2 > 0$ such that $\|x' - x(\lambda_0)\| < \varepsilon_1$ if $\|q - q'\|_\infty < \varepsilon_2$. In conclusion, if $\|(b,q)-(b',q')\| < \varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, from (2.2) we have that $x' \in U \cap \mathbf{F}^{-1}(b',q')$.

$(vi) \Rightarrow (i)$: Note that the lower semicontinuity of $\mathbf{F}^{-1}$ at (b,q) obviously yields $(b,q) \in \operatorname{int} \operatorname{rge} \mathbf{F}$, hence the metric regularity of $\mathbf{F}$ follows from Robinson-Ursescu Theorem. □

In the remaining part of this section we derive an explicit formula for the modulus of metric regularity of the mapping $\mathbf{F}$ defined in (1.7) at $\bar{x}$ for (b,q). We relate $\operatorname{reg}\mathbf{F}(\bar{x}\,|\,(b,q))$ with the modulus of metric regularity of the mappings associated with each one of the linear inequalities whose coefficient vectors belong to $E(b,q)$. In further lines we refer to the sets

$$N(b) := \operatorname{cone}\left\{\begin{pmatrix} a(t) \\ b(t) \end{pmatrix},\ t \in T\right\} \quad \text{and} \quad L(q) := \operatorname{span}\left\{\begin{pmatrix} p(s) \\ q(s) \end{pmatrix},\ s \in S\right\}.$$

Lemma 2.2. *Let $(b,q) \in \mathcal{C}(T, I\!R) \times I\!R^S$. Then*

(i) *The set $E(b,q)$ is compact;*

(ii) *If the constraint system (1.6) satisfies the SCQ condition, then the associated cones*

$$N(b) + L(q) \qquad \text{and} \qquad N(b) + L(q) + \operatorname{cone}\left\{\begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}$$

are closed.

Proof. (i) Recall that an element of $E(b,q)$ has the form

$$\sum_{t\in T} \lambda(t) \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} + \sum_{s \in S} \mu(s) \begin{pmatrix} p(s) \\ q(s) \end{pmatrix},$$

with $\lambda \in I\!\!R_+^{(T)}$, $\mu \in I\!\!R^S$, $\sum_{t\in T} \lambda(t) + \sum_{s\in S} |\mu(s)| = 1$. By Carathéodory's theorem we can assume that the cardinality of the support set of λ is not greater than $n+1$. Denote the corresponding values of λ by $\lambda_i, i = 1, \cdots, n+1$. The set

$$K := \Big\{ (\lambda_1, \cdots, \lambda_{n+1}, \mu_1, \cdots, \mu_m) \in I\!\!R_+^{n+1} \times I\!\!R^m \mid \sum_{i=1}^{n+1} \lambda_i + \sum_{j=1}^{m} |\mu_j| = 1 \Big\}$$

is a compact subset of $I\!\!R^{n+1+m}$. Consider the function

$$f : K \times \left\{ \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, t \in T \right\}^{n+1} \to I\!\!R^{n+1}$$

given by

$$f\left(\lambda_1, \cdots, \lambda_{n+1}, \mu_1, \cdots, \mu_m; \begin{pmatrix} a(t_1) \\ b(t_1) \end{pmatrix}, \cdots, \begin{pmatrix} a(t_{n+1}) \\ b(t_{n+1}) \end{pmatrix}\right) = \sum_{i=1}^{n+1} \lambda_i \begin{pmatrix} a(t_i) \\ b(t_i) \end{pmatrix} + \sum_{j=1}^{m} \mu_j \begin{pmatrix} p(s_j) \\ q(s_j) \end{pmatrix},$$

where we write $\{s_1, \cdots, s_m\} := S$. Clearly, f is a continuous function defined on a compact set and its image is $E(b,q)$. Hence, $E(b,q)$ is compact.

(ii) Since $\{\binom{a(t)}{b(t)},\ t \in T\}$ is a compact set, its convex hull is also compact, and the existence of a Slater element guarantees that 0_{n+1} does not belong to the convex hull. Then, $N(b)$ is closed according to [23, Corollary 9.6.1]. Moreover, the existence of a Slater element yields $N(b) \cap L(q) = \{0_{n+1}\}$, and then [23, Corollary 9.1.3] applies to obtain that $N(b) + L(q)$ is closed. Finally, we apply [8, Theorem 5.3(ii)] to conclude that the associated characteristic cone is closed as well. ◻

Recall that the Farkas Lemma applied to the linear semi-infinite system (1.6) says that the inequality $a^\mathsf{T} x \geq b$ is a consequence of the system (1.6) (when consistent) if and only if

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cl}\left(N(b) + L(q) + \operatorname{cone}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \right).$$

Lemma 2.3. *Let $(\bar{x}, (b,q)) \in \operatorname{gph} \mathbf{F}$ and assume that $\mathbf{F}$ is metrically regular at $\bar{x}$ for (b,q). Then, given $z \in I\!\!R^n$, one has*

$$d\left(z, \mathbf{F}^{-1}(b,q)\right) = \sup_{\binom{u}{v} \in N(b)+L(q)} d\left(z, H(u,v)\right), \tag{2.3}$$

where $H(u,v) := \left\{ x \in I\!\!R^n \mid u^\mathsf{T} x \geq v \right\}$. Moreover, when $z \notin \mathbf{F}^{-1}(b,q)$, the supremum is attained at $(\bar{u}, \bar{u}^\mathsf{T} \hat{z})$, where $\hat{z}$ is a projection of z on $\mathbf{F}^{-1}(b,q)$ and $\bar{u}$ satisfies $\|\bar{u}\|_* = 1$, and

(i) *$\bar{u}^\mathsf{T} x \geq \bar{u}^\mathsf{T} \hat{z}$ for all $x \in \mathbf{F}^{-1}(b,q)$;*

(ii) $\bar{u}^{\mathsf{T}}(\hat{z}-z)=\|\hat{z}-z\|$.

Proof. First, consider the case $z\in\mathbf{F}^{-1}(b,q)$. If $\binom{u}{v}\in N(b)+L(q)$, there exist $\lambda\in\mathbb{R}_{+}^{(T)}$ and $\mu\in\mathbb{R}^{S}$ such that

$$\begin{pmatrix}u\\v\end{pmatrix}=\sum_{t\in T}\lambda(t)\begin{pmatrix}a(t)\\b(t)\end{pmatrix}+\sum_{s\in S}\mu(s)\begin{pmatrix}p(s)\\q(s)\end{pmatrix}.$$

Multiplying by $\binom{z}{-1}$ we obtain $z\in H(u,v)$, and the aimed equality holds trivially. Assume $z\notin\mathbf{F}^{-1}(b,q)$. In order to prove the inequality $\geq$ in (2.3), observe that for all $\binom{u}{v}\in N(b)+L(q)$, the inequality $u^{\mathsf{T}}x\geq v$ is a consequence of the consistent constraint system (1.6). Then $\mathbf{F}^{-1}(b,q)\subset H(u,v)$ which yields $d(z,\mathbf{F}^{-1}(b,q))\geq d(z,H(u,v))$.

To prove the opposite inequality we use the following characterization of projections (for the "if" part, see [17], p. 136, while the "only if" part can be found in [1], Lemma 9). Let A be a non-empty closed convex set in $\mathbb{R}^{n}$, which is endowed with an arbitrary norm $\|\cdot\|$, and let $z\in\mathbb{R}^{n}\setminus A$. Then, $\hat{z}\in A$ is a projection (best approximation) in $\mathbb{R}^{n}$ of z in A (i.e., $\|\hat{z}-z\|=d(z,A)$) if and only if there exists a vector $\bar{u}\in\mathbb{R}^{n}$, which satisfies the conditions (i) and (ii) in the statement of the lemma.

Let $\hat{z}\in\mathbb{R}^{n}$ be a projection of z in $\mathbf{F}^{-1}(b,q)$ and let $\bar{u}$, $\|\bar{u}\|_{*}=1$, satisfy the conditions (i) and (ii) in the statement of the lemma. Appealing to the Farkas Lemma and the closedness of the characteristic cone (according to Lemma 2.2), we write

$$\begin{pmatrix}\bar{u}\\\bar{u}^{\mathsf{T}}\hat{z}\end{pmatrix}=\sum_{t\in T}\lambda(t)\begin{pmatrix}a(t)\\b(t)\end{pmatrix}+\sum_{s\in S}\mu(s)\begin{pmatrix}p(s)\\q(s)\end{pmatrix}+\gamma\begin{pmatrix}0_n\\-1\end{pmatrix},$$

for certain $\lambda\in\mathbb{R}_{+}^{(T)}$, $\mu\in\mathbb{R}^{S}$ and $\gamma\geq 0$. Multiplying both sides in this equality by $\binom{\hat{z}}{-1}$ we conclude that $\gamma=0$, and hence

$$\begin{pmatrix}\bar{u}\\\bar{u}^{\mathsf{T}}\hat{z}\end{pmatrix}\in N(b)+L(q).$$

Moreover,

$$d\left(z,H\left(\bar{u},\bar{u}^{\mathsf{T}}\hat{z}\right)\right)=\frac{|\bar{u}^{\mathsf{T}}(z-\hat{z})\,|}{\|\bar{u}\|_{*}}=\|z-\hat{z}\|=d\left(z,\mathbf{F}^{-1}(b,q)\right).$$

Thus, we obtain

$$d\left(z,\mathbf{F}^{-1}(b,q)\right)\leq\sup_{\binom{u}{v}\in N(b)+L(q)}d\left(z,H(u,v)\right)$$

and the proof is complete. □

The following theorem provides a formula for $\operatorname{reg}\mathbf{F}(\bar{x}\,|\,(b,q))$. We adopt the conventions $\inf\emptyset=\infty$, $1/\infty=0$ and $1/0=\infty$ and use the notation of Example 1.2.

Theorem 2.4. *Let* $(\bar{x},(b,q)) \in \operatorname{gph}\mathbf{F}$. *Then,*

$$\operatorname{reg}\mathbf{F}(\bar{x}\,|\,(b,q)) = \sup_{\binom{u}{v}\in E(b,q)} \operatorname{reg} F_{\binom{u}{v}}(\bar{x}\,|\,0). \tag{2.4}$$

Proof. If $\mathbf{F}$ is not metrically regular at $\bar{x}$ for (b,q), then $\operatorname{reg}\mathbf{F}(\bar{x}\,|\,(b,q)) = \infty$ and, at the same time, $0_{n+1} \in E\,(b,q)$ by Theorem 2.1. Then, using the formula (1.2) we have

$$\sup_{\binom{u}{v}\in E(b,q)} \operatorname{reg} F_{\binom{u}{v}}(\bar{x}\,|\,0) = \operatorname{reg} F_{\binom{0_n}{0}}(\bar{x}\,|\,0) = \infty.$$

Assume that $\mathbf{F}$ is metrically regular at $\bar{x}$ for (b,q).

Proof of the inequality $\leq$ *in (2.4).* On the contrary, assume that there exists $\alpha > 0$ such that

$$\operatorname{reg}\mathbf{F}(\bar{x}\,|\,(b,q)) > \alpha > \sup_{\binom{u}{v}\in E(b,q)} \operatorname{reg} F_{\binom{u}{v}}(\bar{x}\,|\,0). \tag{2.5}$$

The first inequality in (2.5) yields that there exist sequences $\{x^r\}$ converging to $\bar{x}$ and $\{(b^r,q^r)\}$ converging to (b,q) such that, for each r,

$$d\left(x^r, \mathbf{F}^{-1}\left(b^r,q^r\right)\right) > \alpha\, d\left(\left(b^r,q^r\right), \mathbf{F}\left(x^r\right)\right). \tag{2.6}$$

Note that, by the metric regularity assumption, for r large enough (say $r \geq r_0$) one has $(b^r,q^r) \in \operatorname{int}\operatorname{rge}\mathbf{F}$, and $\mathbf{F}$ will be metrically regular at every feasible point of (1.6) with right-hand side (b^r,q^r). Then, since under the current hypotheses $x^r \notin \mathbf{F}^{-1}\left(b^r,q^r\right)$, Lemma 2.3 yields

$$d\left(x^r, \mathbf{F}^{-1}\left(b^r,q^r\right)\right) = d\left(x^r, H(u^r,(u^r)^{\mathsf{T}}\hat{x}^r)\right), \tag{2.7}$$

where $\hat{x}^r$ is a projection of x^r on $\mathbf{F}^{-1}(b^r,q^r)$ and

$$\begin{pmatrix} u^r \\ \left(u^r\right)^{\mathsf{T}}\hat{x}^r \end{pmatrix} \in N\left(b^r\right) + L\left(q^r\right),$$

where u^r, with $\|u^r\|_* = 1$, satisfies the conditions (i) and (ii) in the statement of Lemma 2.3. Write

$$\begin{pmatrix} u^r \\ \left(u^r\right)^{\mathsf{T}}\hat{x}^r \end{pmatrix} = \sum_{t\in T}\lambda^r(t)\begin{pmatrix} a(t) \\ b^r(t) \end{pmatrix} + \sum_{s\in S}\mu^r(s)\begin{pmatrix} p(s) \\ q^r(s) \end{pmatrix} \tag{2.8}$$

for some $\{\lambda^r\} \subset \mathbb{R}_+^{(T)}$ and $\{\mu^r\} \subset \mathbb{R}^S$.

We organize the rest of the proof of the inequality $\leq$ in (2.4) in four steps.

Step 1. Denoting

$$\gamma_r := \sum_{t\in T}\lambda^r(t) + \sum_{s\in S}|\mu^r(s)|$$

we will prove that

$$\alpha < \gamma_r \quad \text{for all} \quad r \geq r_0.$$

Multiplying both sides in (2.8) by $\binom{-x^r}{1}$, we have

$$\begin{aligned}
&\begin{pmatrix} u^r \\ (u^r)^\mathsf{T}\hat{x}^r \end{pmatrix}^\mathsf{T} \begin{pmatrix} -x^r \\ 1 \end{pmatrix} = \sum_{t\in T} \lambda^r(t)(b^r(t) - a(t)^\mathsf{T} x^r) + \sum_{s\in S} \mu^r(s)(q^r(s) - p(s)^\mathsf{T} x^r) \\
&\leq (\sum_{t\in T} \lambda^r(t)) \sup_{t\in T} \left[b^r(t) - a(t)^\mathsf{T} x^r\right]_+ + (\sum_{s\in S} |\mu^r(s)|) \sup_{s\in S} |q^r(s) - p(s)^\mathsf{T} x^r| \\
&\leq (\sum_{t\in T} \lambda^r(t) + \sum_{s\in S} |\mu^r(s)|) d((b^r, q^r), \mathbf{F}(x^r)),
\end{aligned}$$

also taking into account that

$$d\left(\left(b^r, q^r\right), \mathbf{F}\left(x^r\right)\right) = \max\left\{\sup_{t\in T}[b^r(t) - a(t)^\mathsf{T} x^r]_+, \sup_{s\in S} |q^r(s) - p(s)^\mathsf{T} x^r|\right\}; \tag{2.9}$$

that is,

$$d\left(\left(b^r, q^r\right), \mathbf{F}\left(x^r\right)\right) \geq \gamma_r^{-1} \left(u^r\right)^\mathsf{T} \left(\hat{x}^r - x^r\right).$$

Hence, having in mind (2.6) and (2.7), we obtain

$$\frac{\left(u^r\right)^\mathsf{T} \left(\hat{x}^r - x^r\right)}{\|u^r\|_*} = d\left(x^r, H(u^r, (u^r)^\mathsf{T} \hat{x}^r)\right) > \alpha\gamma_r^{-1} \left(u^r\right)^\mathsf{T} \left(\hat{x}^r - x^r\right)$$

which yields

$$\alpha\gamma_r^{-1} < 1, \quad \text{for all } r \geq r_0.$$

Step 2. The sequence $\{\hat{x}^r\}$ converges to $\bar{x}$.

The lower semicontinuity of $\mathbf{F}^{-1}$ at (b, q) (Theorem 2.1) yields the existence of $\{z^r\}$ converging to $\bar{x}$, with $z^r \in \mathbf{F}^{-1}\left(b^r, q^r\right)$ for all $r \geq r_0$. Hence, by the triangle inequality,

$$\|\bar{x} - \hat{x}^r\| \leq \|\bar{x} - x^r\| + \|x^r - \hat{x}^r\| \leq \|\bar{x} - x^r\| + \|x^r - z^r\| \to 0 \text{ as } r \to \infty,$$

hence the conclusion.

Step 3. The sequence $\{\gamma_r\}$ is bounded.

On the contrary, assume that $\gamma_r \to \infty$ as $r \to \infty$ (for some subsequence). Then

$$\begin{aligned}
0_{n+1} &= \lim_{r\to\infty} \frac{1}{\gamma_r} \begin{pmatrix} u^r \\ \left(u^r\right)^\mathsf{T} \hat{x}^r \end{pmatrix} = \lim_{r\to\infty} \sum_{t\in T} \frac{\lambda^r(t)}{\gamma_r} \begin{pmatrix} a(t) \\ b^r(t) \end{pmatrix} + \sum_{s\in S} \frac{\mu^r(s)}{\gamma_r} \begin{pmatrix} p(s) \\ q^r(s) \end{pmatrix} \\
&= \lim_{r\to\infty} \left\{\sum_{t\in T} \frac{\lambda^r(t)}{\gamma_r} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} + \sum_{s\in S} \frac{\mu^r(s)}{\gamma_r} \begin{pmatrix} p(s) \\ q(s) \end{pmatrix}\right\} \\
&+ \lim_{r\to\infty} \left\{\sum_{t\in T} \frac{\lambda^r(t)}{\gamma_r} \begin{pmatrix} 0_n \\ b^r(t) - b(t) \end{pmatrix} + \sum_{s\in S} \frac{\mu^r(s)}{\gamma_r} \begin{pmatrix} 0_n \\ q^r(s) - q(s) \end{pmatrix}\right\}.
\end{aligned}$$

The last limit is equal to zero, hence, by the closedness of $E\,(b,q)$ (Lemma 2.2) we obtain $0_{n+1} \in E\,(b,q)$ contradicting, via Theorem 2.1, the assumed metric regularity.

Step 4. Completion of the proof of $\leq$ in (2.4).

Since $\{\gamma_r\}$ is bounded, we may assume without loss of generality that it converges to certain $\gamma \geq \alpha > 0$ (see Step 1). Also, since $\|u^r\|_* = 1$, we assume that $u^r \to u$ as $r \to \infty$. Thus, taking into account (2.8) we obtain

$$\frac{1}{\gamma}\begin{pmatrix} u \\ u^\top \bar{x} \end{pmatrix} = \lim_{r\to\infty} \frac{1}{\gamma_r}\begin{pmatrix} u^r \\ (u^r)^\top \hat{x}^r \end{pmatrix} \in E\,(b,q)\,,$$

by using the closedness of $E\,(b,q)$ and following an argument similar to the proof of Step 3. In such a way, we have

$$\frac{1}{\gamma}\begin{pmatrix} u \\ u^\top \bar{x} \end{pmatrix} \in E\,(b,q) \;\text{ with }\; \frac{1}{\gamma}u \neq 0_n \;\text{ and }\; \gamma \geq \alpha.$$

Therefore, according to (1.2) we have

$$\operatorname{reg} F_{\frac{1}{\gamma}\binom{u}{u^\top \bar{x}}}\,(\bar{x} \mid 0) = \|u/\gamma\|_*^{-1} = \gamma \geq \alpha,$$

which contradicts the assumption (2.5). Hence, we have $\leq$ in (2.4).

Proof of the inequality $\geq$ in (2.4).

On the contrary, suppose that there exists $\alpha > 0$ such that

$$\text{(2.10)} \qquad \operatorname{reg} \mathbf{F}(\bar{x}\,|\,(b,q)) < \alpha < \sup_{\binom{u}{v}\in E(b,q)} \operatorname{reg} F_{\binom{u}{v}}\,(\bar{x}\,|\,0)\,.$$

Then there exist $\binom{\bar{u}}{\bar{v}} \in E\,(b,q)$ with $\bar{u}^\top \bar{x} = \bar{v}$ (otherwise $\operatorname{reg} F_{\binom{\bar{u}}{\bar{v}}}\,(\bar{x}\,|\,0) = 0$), and two sequences $\{x^r\}$ converging to $\bar{x}$ and $\{\beta_r\}$ converging to 0 such that

$$\text{(2.11)} \qquad d\left(x^r, F^{-1}_{\binom{\bar{u}}{\bar{v}}}\,(\beta_r)\right) > \alpha\, d\left(\beta_r, F_{\binom{\bar{u}}{\bar{v}}}\,(x^r)\right), \quad \text{for } r = 1, 2, \cdots.$$

Next, we construct two sequences, $\{b^r\} \subset \mathcal{C}\,(T, I\!R)$ and $\{q^r\} \subset I\!R^S$ such that $\{(b^r, q^r)\}$ converges to (b,q),

$$\text{(2.12)} \qquad d\left(x^r, \mathbf{F}^{-1}\,(b^r, q^r)\right) \geq d\left(x^r, F^{-1}_{\binom{\bar{u}}{\bar{v}}}\,(\beta_r)\right) \quad \text{for all } \; r,$$

and

$$\text{(2.13)} \qquad d\left((b^r, q^r)\,, \mathbf{F}\,(x^r)\right) = d\left(\beta_r, F_{\binom{\bar{u}}{\bar{v}}}\,(x^r)\right) \quad \text{for all } \; r.$$

The first inequality of (2.10) implies that for r large enough

$$d\left(x^r, \mathbf{F}^{-1}\left(b^r, q^r\right)\right) \leq \alpha\, d\left(\left(b^r, q^r\right), \mathbf{F}\left(x^r\right)\right), \tag{2.14}$$

and then the combination of (2.11), (2.12), (2.13) and (2.14) leads to absurd, inasmuch for r large enough,

$$\begin{aligned} \alpha d\left(\left(b^r, q^r\right), \mathbf{F}\left(x^r\right)\right) &\geq d\left(x^r, \mathbf{F}^{-1}\left(b^r, q^r\right)\right) \geq d\left(x^r, F^{-1}_{\binom{\bar{u}}{\bar{v}}}\left(\beta_r\right)\right) \\ &> \alpha\, d\left(\beta_r, F_{\binom{\bar{u}}{\bar{v}}}\left(x^r\right)\right) = \alpha d\left(\left(b^r, q^r\right), \mathbf{F}\left(x^r\right)\right). \end{aligned}$$

Write

$$\binom{\bar{u}}{\bar{v}} = \sum_{i\in I} \lambda_i \binom{a\left(t_i\right)}{b\left(t_i\right)} + \sum_{j\in J} \mu_j \binom{p\left(s_j\right)}{q\left(s_j\right)}, \tag{2.15}$$

with $\lambda_i > 0$, $i \in I$, $\mu_j \neq 0$, $j \in J$, and $\sum_{i\in I} \lambda_i + \sum_{j\in J} |\mu_j| = 1$, where I and J are finite index sets, possibly (but not simultaneously) empty. Multiplying both sides in (2.15) by $\binom{\bar{x}}{-1}$ we have

$$0 = \bar{u}^\top \bar{x} - \bar{v} = \sum_{i\in I} \lambda_i \left(a\left(t_i\right)^\top \bar{x} - b\left(t_i\right)\right).$$

Since $\lambda_i > 0$ and $a\left(t_i\right)^\top \bar{x} - b\left(t_i\right) \geq 0$ for all $i \in I$, we obtain

$$a(t_i)^\top \bar{x} = b(t_i), \quad i \in I.$$

We will obtain (2.12) and (2.13) if $\{b^r\}$ and $\{q^r\}$ are constructed in such a way that

$$\begin{array}{lll} b^r(t) \leq \beta_r + \bar{v} + (a(t) - \bar{u})^\top x^r & \text{for all} & t \in T, \\ |q^r(s) - p(s)^\top x^r| \leq \beta_r + \bar{v} - \bar{u}^\top x^r & \text{for all} & s \in S, \end{array} \tag{2.16}$$

and

$$\begin{array}{ll} b^r\left(t_i\right) = \beta_r + \bar{v} + \left(a\left(t_i\right) - \bar{u}\right)^\top x^r & \text{for all } i \in I, \\ q^r\left(s_j\right) = p\left(s_j\right)^\top x^r + \left(\operatorname{sign} \mu_j\right)\left(\beta_r + \bar{v} - \bar{u}^\top x^r\right) & \text{for all } j \in J. \end{array} \tag{2.17}$$

Note that, since $x^r \notin F^{-1}_{\binom{\bar{u}}{\bar{v}}}\left(\beta_r\right)$ from (2.11), one has $\beta_r + \overline{v} - \bar{u}^\top x^r > 0$ for all r. More precisely, from (2.9) and the fact that $d(\beta_r, F_{\binom{\bar{u}}{\bar{v}}}(x^r)) = \beta_r + \bar{v} - \bar{u}^\top x^r$, (2.16) yields

$$d\left(\left(b^r, q^r\right), \mathbf{F}\left(x^r\right)\right) \leq d\left(\beta_r, F_{\binom{\bar{u}}{\bar{v}}}\left(x^r\right)\right).$$

The equality then comes from (2.17).

In order to establish (2.12) we will show that

$$\sum_{i\in I}\lambda_i\begin{pmatrix}a\left(t_i\right)\\ b^r\left(t_i\right)\end{pmatrix}+\sum_{j\in J}\mu_j\begin{pmatrix}p\left(s_j\right)\\ q^r\left(s_j\right)\end{pmatrix}=\begin{pmatrix}\bar{u}\\ \bar{v}+\beta_r\end{pmatrix}. \tag{2.18}$$

Indeed, $\sum_{i\in I}\lambda_i a\left(t_i\right)+\sum_{j\in J}\mu_j p\left(s_j\right)=\bar{u}$ comes from (2.15). Moreover, applying (2.17), we have

$$\begin{aligned}
&\sum_{i\in I}\lambda_i b^r\left(t_i\right)+\sum_{j\in J}\mu_j q^r\left(s_j\right)\\
&=\sum_{i\in I}\lambda_i a\left(t_i\right)^{\top}x^r+\sum_{i\in I}\lambda_i\left(\beta_r+\bar{v}-\bar{u}^{\top}x^r\right)\\
&+\sum_{j\in J}\mu_j p\left(s_j\right)^{\top}x^r+\sum_{j\in J}\mu_j\left(\operatorname{sign}\mu_j\right)\left(\beta_r+\bar{v}-\bar{u}^{\top}x^r\right)\\
&=\left(\sum_{i\in I}\lambda_i a\left(t_i\right)+\sum_{j\in J}\mu_j p\left(s_j\right)\right)^{\top}x^r+\left(\sum_{i\in I}\lambda_i+\sum_{j\in J}\left|\mu_j\right|\right)\left(\beta_r+\bar{v}-\bar{u}^{\top}x^r\right)\\
&=\bar{u}^{\top}x^r+\beta_r+\bar{v}-\bar{u}^{\top}x^r=\beta_r+\bar{v}.
\end{aligned}$$

The equality (2.18) together with Farkas Lemma yield that $\bar{u}^{\top}x\geq\beta_r+\bar{v}$ is a consequence of the system $\{a(t)^{\top}x\geq b^r(t),\ t\in T;\ p(s)^{\top}x=q^r(s),\ s\in S\}$, and hence (2.12) is guaranteed.

We now construct (b^r,q^r) verifying (2.16) and (2.17). We know that $a(t)^{\top}\bar{x}\geq b(t)$, for all $t\in T$, $a\left(t_i\right)^{\top}\bar{x}=b\left(t_i\right),\ i\in I$, and $p(s)^{\top}\bar{x}=q(s)$, for all $s\in S$. Observe that, taking into account that $\bar{u}^{\top}\bar{x}=\bar{v}$, we have

$$\begin{aligned}
&|(\beta_r+\bar{v}+(a(t)-\bar{u})^{\top}x^r)-a(t)^{\top}\bar{x}|\\
&\leq|\beta_r|+|\bar{u}^{\top}(\bar{x}-x^r)|+|a(t)^{\top}(x^r-\bar{x})|\\
&\leq|\beta_r|+\|\bar{u}\|_*\|x^r-\bar{x}\|+\sup\nolimits_{t\in T}\|a(t)\|_*\|x^r-\bar{x}\|=:\varepsilon_r^1,
\end{aligned}$$

and

$$\begin{aligned}
&|(p(s)^{\top}x^r\pm(\beta_r+\bar{v}-\bar{u}^{\top}x^r))-p(s)^{\top}\bar{x}|\\
&\qquad\leq|\beta_r|+\|\bar{u}\|_*\|x^r-\bar{x}\|+\max\nolimits_{s\in S}\|p(s)\|_*\|x^r-\bar{x}\|=:\varepsilon_r^2.
\end{aligned} \tag{2.19}$$

Let $\varepsilon_r:=\max\{\varepsilon_r^1,\varepsilon_r^2\}$ for all r, then $\lim_r\varepsilon_r=0$. The sequence $\{b^r\}$ is constructed as follows. From the preceding inequalities, we have

$$a(t)^{\top}\bar{x}-\varepsilon_r\leq\beta_r+\bar{v}+(a(t)-\bar{u})^{\top}x^r\leq a(t)^{\top}\bar{x}+\varepsilon_r\quad\text{for all } t\in T. \tag{2.20}$$

Let

$$A:=\left\{t\in T\mid a(t)^{\top}\bar{x}-\varepsilon_r\leq b(t)\leq a(t)^{\top}\overline{x}\right\}\supset\left\{t_i,i\in I\right\},$$

and

$$B := \left\{ t \in T \mid a(t)^\mathsf{T}\bar{x} - 2\varepsilon_r \geq b(t) \right\}.$$

It is clear that A and B are disjoint compact (possibly empty) subsets of T. By using Urysohn's Lemma, we find $\varphi \in \mathcal{C}(T, [0,1])$ such that

$$\varphi(t) = \begin{cases} 1, & \text{if } \ t \in A, \\ 0, & \text{if } \ t \in B. \end{cases}$$

Define

$$b^r(t) := \left(\beta_r + \bar{v} + (a(t) - \bar{u})^\mathsf{T} x^r\right)\varphi(t) + b(t)\left(1 - \varphi(t)\right). \tag{2.21}$$

If $t \in B,$ then $b^r(t) = b(t)$. If $t \notin B$, one has

$$b(t) \in (a(t)^\mathsf{T}\overline{x} - 2\varepsilon_r, a(t)^\mathsf{T}\bar{x}]$$

and

$$\beta_r + \bar{v} + (a(t) - \bar{u})^\mathsf{T} x^r \in \left[a(t)^\mathsf{T}\bar{x} - \varepsilon_r, a(t)^\mathsf{T}\bar{x} + \varepsilon_r\right].$$

Hence (2.21) implies

$$b^r(t) \in \left(a(t)^\mathsf{T}\bar{x} - 2\varepsilon_r, a(t)^\mathsf{T}\bar{x} + \varepsilon_r\right],$$

which in turn yields

$$|b^r(t) - b(t)| \leq 3\varepsilon_r, \quad \text{for all } \ t \in T; \quad \text{that is,} \quad \|b^r - b\|_\infty \leq 3\varepsilon_r.$$

Moreover, for $t \in A$ (in particular for $t = t_i$, $i \in I$) we have

$$b^r(t) = \beta_r + \bar{v} + (a(t) - \bar{u})^\mathsf{T} x^r.$$

For $t \in B$ one obtains

$$b^r(t) = b(t) \leq a(t)^\mathsf{T}\bar{x} - 2\varepsilon_r \leq a(t)^\mathsf{T}\overline{x} - \varepsilon_r \leq \beta_r + \bar{v} + (a(t) - \bar{u})^\mathsf{T} x^r.$$

Finally, for $t \in T \setminus (A \cup B)$ we have

$$b(t) < a(t)^\mathsf{T}\bar{x} - \varepsilon_r \leq \beta_r + \bar{v} + (a(t) - \overline{u})^\mathsf{T} x^r,$$

and then (2.21) yields

$$b^r(t) \leq \beta_r + \bar{v} + (a(t) - \bar{u})^\mathsf{T} x^r.$$

Therefore, the first set of conditions in (2.16) and (2.17) holds. To complete the construction, we define q^r in the following way:

$$q^r(s) = \begin{cases} p\,(s_j)^\mathsf{T}\, x^r + (\operatorname{sign}\mu_j)\left(\beta_r + \bar{v} - \bar{u}^\mathsf{T} x^r\right), & \text{if} \quad s = s_j, \quad j \in J, \\ p(s)^\mathsf{T} x^r + \left(\beta_r + \bar{v} - \overline{u}^\mathsf{T} x^r\right), & \text{if} \quad s \in S \setminus \{s_j, j \in J\}\,. \end{cases}$$

So, (2.19) gives us $|q^r(s) - q(s)| \leq \varepsilon_r$ for all $s \in S$ and, moreover,

$$|q^r(s) - p(s)^\mathsf{T} x^r| = \beta_r + \bar{v} - \bar{u}^\mathsf{T} x^r \quad \text{for} \quad s \in S.$$

Hence, $\{(b^r, q^r)\}$ converges to (b, q) and also the second conditions in (2.16) and (2.17) hold, which completes the proof. □

From the theorem just proved and the formula (1.2) we have

Corollary 2.5. *Let* $(\bar{x}, (b, q)) \in \operatorname{gph} \boldsymbol{F}$ *. Then*

$$\operatorname{reg} \mathbf{F}(\bar{x}\,|\,(b, q)) = \left(\inf \left\{ \, \|u\|_* \mid \begin{pmatrix} u \\ u^\mathsf{T} \bar{x} \end{pmatrix} \in E(b, q) \, \right\} \right)^{-1}.$$

3 Radius Theorem and Distance to Infeasibility

In this section we first prove a radius theorem of the form of the equality (1.5) for the mapping $\mathbf{F}$ defined in (1.7). The radius of metric regularity of this mapping is defined as

$$\operatorname{rad} \mathbf{F}(\bar{x}\,|\,(b, q)) = \inf_{l \in L\left(I\!\!R^n, \mathcal{C}(T, I\!\!R) \times I\!\!R^S\right)} \{\|l\| \mid \mathbf{F} + l \ \text{ not metrically regular at } \bar{x} \text{ for } (b, q) + l(\bar{x}) \ \}\,.$$

We write

$$l\,(x) = \begin{pmatrix} g(\cdot)^\mathsf{T} x \\ h(\cdot)^\mathsf{T} x \end{pmatrix} \quad \text{for} \quad g \in \mathcal{C}\,(T, I\!\!R^n) \ \text{ and } \ h : S \to I\!\!R^n,$$

and use the usual operator norm

$$\|l\| = \max \left\{ \sup_{t \in T} \|g(t)\|_*\,, \max_{s \in S} \|h(s)\|_* \right\}.$$

Theorem 3.1. *For any* $(\bar{x}, (b, q)) \in \operatorname{gph} \mathbf{F}$,

$$\operatorname{rad} \mathbf{F}(\bar{x}\,|\,(b, q)) = \frac{1}{\operatorname{reg} \mathbf{F}(\bar{x}\,|\,(b, q))}. \tag{3.1}$$

Proof. The general Theorem 1.4 gives the inequality $\geq$, thus it is enough to establish $\leq$. This inequality is trivially satisfied if $\mathbf{F}$ is not metrically regular at $\bar{x}$ for (b, q). Assume that $\mathbf{F}$ is metrically regular. According to Corollary 2.5 we have

$$\frac{1}{\operatorname{reg} \mathbf{F}(\bar{x}\,|\,(b, q))} = \inf \left\{ \|u\|_* \mid \begin{pmatrix} u \\ u^\mathsf{T} \bar{x} \end{pmatrix} \in E(b, q) \right\}. \tag{3.2}$$

Let us discuss first the easy case when the set

$$D := \left\{ u \mid \begin{pmatrix} u \\ u^\top \bar{x} \end{pmatrix} \in E(b,q) \right\}$$

is empty. In this case the only possibility is that S is empty and that $\bar{x}$ is a Slater element of the system $\{a(t)^\top x \geq b(t),\, t \in T\}$. Obviously, $\bar{x}$ is also a Slater element of a system in the form $\{(a(t)+g(t))^\top x \geq b(t) + g(t)^\top \bar{x},\, t \in T\}$ for any g and then $\operatorname{rad} \mathbf{F}\left(\bar{x} \middle| (b,q)\right) = \infty$. The equality (3.1) then follows from the convention $\inf \emptyset = \infty$.

Consider the case when $D \neq \emptyset$. Then, due to the compactness of $E\,(b,q)$, the right hand side of (3.2) is attained at some $\bar{u}$. Specifically, we have

$$\begin{pmatrix} \bar{u} \\ \bar{u}^\top \bar{x} \end{pmatrix} = \sum_{t \in T} \lambda(t) \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} + \sum_{s \in S} \mu(s) \begin{pmatrix} p(s) \\ q(s) \end{pmatrix}$$

for some $\lambda \in I\!R_+^{(T)}$ and $\mu \in I\!R^S$, with $\sum_{t\in T} \lambda(t) + \sum_{s \in S} |\mu(s)| = 1$. Define $l : I\!R^n \to \mathcal{C}\,(T, I\!R) \times I\!R^S$ as

$$l(x)(t,s) := \begin{pmatrix} -\bar{u}^\top x \\ (-\operatorname{sign} \mu(s)) \bar{u}^\top x \end{pmatrix} \qquad \text{for } t \in T \text{ and } s \in S.$$

Then we have

$$\begin{aligned}
&\sum_{t\in T} \lambda(t) \left(\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} - \begin{pmatrix} \bar{u} \\ \bar{u}^\top \bar{x} \end{pmatrix} \right) + \sum_{s \in S} \mu(s) \left(\begin{pmatrix} p(s) \\ q(s) \end{pmatrix} - (\operatorname{sign} \mu(s)) \begin{pmatrix} \bar{u} \\ \bar{u}^\top \bar{x} \end{pmatrix} \right) \\
&= \left\{ \sum_{t\in T} \lambda(t) \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} + \sum_{s\in S} \mu(s) \begin{pmatrix} p(s) \\ q(s) \end{pmatrix} \right\} - \left(\sum_{t \in T} \lambda(t) + \sum_{s \in S} |\mu(s)| \right) \begin{pmatrix} \bar{u} \\ \bar{u}^\top \bar{x} \end{pmatrix} \\
&= \begin{pmatrix} \bar{u} \\ \bar{u}^\top \bar{x} \end{pmatrix} - \begin{pmatrix} \overline{u} \\ \bar{u}^\top \bar{x} \end{pmatrix} = 0_{n+1}.
\end{aligned}$$

According to Theorem 2.1, the last equality means that $\mathbf{F} + l$ is not metrically regular at $\bar{x}$ for $(b,q) + l\,(\bar{x})$. Then

$$\operatorname{rad} \mathbf{F}\left(\bar{x} \middle| (b,q)\right) \leq \|l\| = \|\bar{u}\|_* = \frac{1}{\operatorname{reg} \mathbf{F}\left(\bar{x} \middle| (b,q)\right)}$$

and hence we obtain the equality (3.1). ◻

As an application of the preceding theorem, in the rest of this section we relate the radius of metric regularity of a mapping of the form (1.7), with $S = \emptyset$ and under certain conditions on the coefficients, to its distance to infeasibility. We consider a consistent system of the form

$$a(t)^\top x \geq b(t) \text{ for all } t \in T \tag{3.3}$$

and use some conditions and results from [1].

First, we assume that the norm $\|\cdot\|$ in $I\!R^{n+1}$ has the following property:

$$\left\|\begin{pmatrix}a\\b\end{pmatrix}\right\| = \left\|\begin{pmatrix}a\\-b\end{pmatrix}\right\| \quad \text{for all} \quad \begin{pmatrix}a\\b\end{pmatrix} \in I\!R^{n+1}.$$

In particular, this property implies

$$\left\|\begin{pmatrix}a\\b_1\end{pmatrix}\right\| \le \left\|\begin{pmatrix}a\\b_2\end{pmatrix}\right\| \quad \text{whenever} \quad |b_1| \le |b_2|.$$

Moreover, in $I\!R^n$ we use the norm $\|a\| := \left\|\binom{a}{0}\right\|$. All these properties are equivalent to the corresponding ones for the dual norms, see [1].

We denote by Θ the set of all the linear inequality systems (3.3) and by Θ_c the subset of Θ containing all consistent systems in Θ. Theorem 6.2 in [7] gives a characterization of the interior of Θ_c in an appropriate topology when all coefficients are perturbed versus the case when only the right hand side is perturbed. In the latter case the system (3.3) is in $\operatorname{int}\Theta_c$ if and only if $b \in \operatorname{int}\operatorname{rge}\mathbf{F}$, where $\mathbf{F}(x) := a(\cdot)^{\mathsf{T}}x - \mathcal{C}\left(T, I\!R_+\right),\ x \in I\!R^n$. Therefore, the system (3.3) is in $\operatorname{int}\Theta_c$ if and only if consistency is preserved under small perturbations of the right-hand-side coefficients $b(t)$.

From the analysis in [1], the distance to infeasibility of a system from Θ_c can be written as

$$(3.4)\qquad \inf_{\binom{g}{\gamma}\in\mathcal{C}\left(T,R^{n+1}\right)} \left\{ \sup_{t\in T} \left\|\begin{pmatrix}g(t)\\\gamma(t)\end{pmatrix}\right\|_* \,\middle|\, \left\{ \left(a(t)+g(t)\right)^{\mathsf{T}} x \ge b(t)+\gamma(t),\ t\in T \right\} \in \Theta\setminus\Theta_c \right\}.$$

The set Θ_c in this formula can be replaced by $\operatorname{int}\Theta_c$. Theorem 6 in [1] provides a formula for the distance to infeasibility in terms of the so-called *hypographical set* defined as

$$(3.5)\qquad H := \operatorname{co}\left\{ \begin{pmatrix}a(t)\\b(t)\end{pmatrix}, \quad t\in T \right\} + I\!R_+\begin{pmatrix}0_n\\-1\end{pmatrix},$$

where

$$I\!R_+\begin{pmatrix}0_n\\-1\end{pmatrix} := \left\{ \begin{pmatrix}0_n\\\lambda\end{pmatrix} : \lambda \le 0 \right\}.$$

Specifically, Theorem 6 in [1] states that the distance to infeasibility coincides with $d_{\|\cdot\|_*}\left(0_{n+1}, H\right)$.

From the definition given in the beginning of this section and by the Robinson-Ursescu theorem (remembering that the system (3.3) is in $\operatorname{int}\Theta_c$ if and only if $b \in \operatorname{int}\operatorname{rge}\mathbf{F}$), the radius of metric regularity $\operatorname{rad}\mathbf{F}\left(\bar{x}\,|\,b\right)$ of the system (3.3) is equal to

$$(3.6)\qquad \inf_{g\in\mathcal{C}(T,R^n)} \left\{ \sup_{t\in T} \|g(t)\|_* \mid \left\{ \left(a(t)+g(t)\right)^{\mathsf{T}} x \ge b(t)+g(t)^{\mathsf{T}}\bar{x},\ t\in T \right\} \in \Theta\setminus\operatorname{int}\Theta_c \right\}.$$

Note that, while the definition of the distance to infeasibility assumes arbitrary perturbations of all the coefficients, the expression for the radius of metric regularity involves specific perturbations of the right-hand-side only (maintaining the feasibility of $\bar{x} \in \mathbb{R}^n$). In fact, the distance to infeasibility (3.4) does not coincide in general with the radius of metric regularity (3.6), as the following example shows:

Example 3.2. Consider the system (3.3) in $\mathbb{R}$, with $a(t) = 1$ and $b(t) = -1$ for all $t \in T$. The distance to infeasibility in the Euclidean norm is $\sqrt{2}$ while $\operatorname{rad} \mathbf{F}(\bar{x}|b) = \infty$ if $\bar{x} > -1$ and $\operatorname{rad} \mathbf{F}(-1|b) = 1$. □

The following corollary shows that for homogeneous systems (where $b(t) = 0,\ t \in T$) both measures of ill-posedness coincide for $\bar{x} = 0_n$.

Corollary 3.3. *The distance to infeasibility of the system* $\{a(t)^\top x \geq 0,\, t \in T\}$ *is equal to* $\operatorname{rad} \mathbf{F}(0_n | 0_T)$ *as defined in (3.6), where* $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathcal{C}(T, \mathbb{R})$ *is given by* $\mathbf{F}(x) := a(\cdot)^\top x - \mathcal{C}(T, \mathbb{R}_+)$.

Proof. According to Theorem 6 in [1] the referred distance to infeasibility is given by $d_{\|\cdot\|_*}(0_{n+1}, H)$, where the set H is as in (3.5) with $b(t) = 0$, and hence

$$d_{\|\cdot\|_*}(0_{n+1}, H) = \inf\{\|a\|_* \mid a \in \operatorname{co}\{a(t),\, t \in T\}\}.$$

The last quantity is the same as $1/\operatorname{reg} \mathbf{F}(0_n \mid 0_T)$, by virtue of Theorem 2.4 and Corollary 2.5, and also coincides with $\operatorname{rad} \mathbf{F}(0_n \mid 0_T)$, as a consequence of Theorem 3.1 (or, alternatively, comes from Theorem 2.9 in [4], since $\mathbf{F}$ is sublinear with closed graph). □

Consider now the nonhomogeneous case (3.3). For this case one can derive a formula for the distance to infeasibility on the basis of [4], Corollary 4.5 and Theorem 4.7, where a more general conic constraint system is considered and a homogenization procedure is proposed to transform a nonhomogeneous conical system into a homogeneous one. For the semi-infinite system (3.3) considered here, the form of the homogenized mapping obtained in [4] is as follows:

$$\widetilde{\mathbf{F}}(x, \alpha) := \begin{cases} a(\cdot)^\top x - \alpha b(\cdot) - \mathcal{C}(T, \mathbb{R}_+) & \text{for } \alpha \geq 0, \\ \emptyset & \text{for } \alpha < 0. \end{cases}$$

This mapping can be used to obtain a formula for the distance to infeasibility in the way shown in [4].

At the end, we point out another way of computing the distance to infeasibility for the nonhomogeneous system (3.3). By inspection, one can check that

$$d_{\|\cdot\|_*}(0_{n+1}, H) = \inf\left\{ \left\| \begin{pmatrix} u \\ [v]_- \end{pmatrix} \right\|_* \,\middle|\, \begin{pmatrix} u \\ v \end{pmatrix} \in C(b) \right\},$$

where H is the hypographical set associated to the system (3.3) and

$$C(b) := \operatorname{co}\left\{ \begin{pmatrix} a(t) \\ b(t) \end{pmatrix},\, t \in T \right\}.$$

This observation combined with Theorem 2.4, Corollary 2.5 and Theorem 3.1, and an application of Theorem 6 in [1] for the distance to infeasibility of a system (3.3) being an element of Θ_c, gives us the following corollary.

Corollary 3.4. *The distance to infeasibility of the system (3.3) is equal to* $\operatorname{rad}\widehat{\mathbf{F}}(0_n|0_T)$*, where the mapping* $\widehat{\mathbf{F}} : I\!R^n \times I\!R \rightrightarrows \mathcal{C}\left(C\left(b\right), I\!R\right)$ *is given by*

$$\widehat{\mathbf{F}}(x,\alpha) := \left(u^\mathsf{T} x + [v]_-\,\alpha\right)_{\binom{u}{v}\in C(b)} - \mathcal{C}\left(C\left(b\right), I\!R_+\right).$$

This corollary provides an alternative way of computing the distance to infeasibility, via the mapping $\widehat{\mathbf{F}}$ which is different from $\widetilde{\mathbf{F}}$ coming from [4], and is related to another homogenization procedure for determining the distance to infeasibility of non-homogeneous systems. Specifically, $\widehat{\mathbf{F}}$ is associated with the homogeneous system

$$u^\mathsf{T} x + [v]_-\,\alpha \geq 0, \qquad \begin{pmatrix} u \\ v \end{pmatrix} \in C\left(b\right),$$

whose distance to infeasibility is the same as the distance to infeasibility for (3.3). It is an open question which of these formulas would be easier to use for practical computations.

4 Nonlinear constraints

The result obtained for linear semi-infinite systems can easily be extended to nonlinear systems of the form

$$\left\{ \begin{array}{ll} f(t,x) \geq b(t) & \text{for all } t \in T, \\ g(s,x) = q(s) & \text{for all } s \in S, \end{array} \right. \tag{4.1}$$

where $f : T \times I\!R^n \to I\!R$ and $g : S \times I\!R^n \to I\!R$ are continuous functions (g for the discrete topology in S), and the T, S, b and q are the same as before. In order to apply the Lyusternik-Graves theorem at a given feasible point $\bar{x}$ for (b,q) and to reduce a system of nonlinear inequalities to linear ones, we need the following assumption: The function f is strictly differentiable at $\bar{x}$ uniformly in $t \in T$, the latter meaning that

$$\limsup_{x,x'\to\bar{x},\, x\neq x'} \sup_{t\in T} \frac{\|f(t,x) - f(t,x') - \nabla_x f(t,\bar{x})(x - x')\|}{\|x - x'\|} = 0,$$

where $\nabla_x f(t,\bar{x})$ is a row vector, and $\nabla_x f(\cdot,\bar{x})$ is continuous in T. Define $\mathbf{f} : I\!R^n \to \mathcal{C}(T, I\!R)$ as

$$(\mathbf{f}(x))(t) = f(t,x) \quad \text{for } t \in T.$$

Then straight from the definition of strict differentiability we obtain that the function $\mathbf{f}$ is strictly differentiable at $\bar{x}$ with a strict derivative mapping $D\mathbf{f}(\bar{x})(t) = \nabla_x f(t,\bar{x})$, $t \in T$, where $L(I\!R^n, \mathcal{C}(T, I\!R))$

is identified with $\mathcal{C}(T, I\!R^n)$ (see the beginning of Section 3). We also assume that the function $g(s, \cdot)$ is strictly differentiable at $\bar{x}$ for every $s \in S$. Along with the system (4.1) consider the mapping

$$\mathbf{N}(x) := \begin{pmatrix} f(\cdot, x) \\ g(\cdot, x) \end{pmatrix} - \begin{pmatrix} \mathcal{C}(T, I\!R_+) \\ 0_S \end{pmatrix}. \tag{4.2}$$

According to the Lyusternik-Graves theorem, Theorem 2.4 and Corollary 2.5, the regularity modulus of the mapping $\mathbf{N}$ is given by the formula

$$\operatorname{reg} \mathbf{N}(\bar{x} \,|\, (b, q)) = \sup_{\binom{u}{v} \in E(\tilde{b}, \tilde{q})} \operatorname{reg} F_{\binom{u}{v}}(\bar{x} \,|\, 0) = \left(\inf \left\{ \|u\|_* \mid \begin{pmatrix} u \\ u^\mathsf{T} \bar{x} \end{pmatrix} \in E\left(\tilde{b}, \tilde{q}\right) \right\} \right)^{-1},$$

where the mapping $F_{\binom{u}{v}}$ is given in Example 1.2 and the set $E(\tilde{b}, \tilde{q})$ is defined in (2.1) for

$$\begin{aligned} &a(t) = \nabla_x f(t, \bar{x})^\mathsf{T}, \quad p(s) = \nabla_x g(s, \bar{x})^\mathsf{T}, \\ &\tilde{b}(t) = b(t) - f(t, \bar{x}) + \nabla_x f(t, \bar{x})\bar{x}, \quad t \in T, \\ &\tilde{q}(s) = q(s) - g(s, \bar{x}) + \nabla_x g(s, \bar{x})\bar{x}, \quad s \in S. \end{aligned}$$

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